Lemmas and proofs for “Implicit Polarized F: local type inference for impredicativity”

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Contents

Definitions

Values	$v ::= x \mid \{t\}$
Computations	$t ::= \lambda x : P.\, t \mid \Lambda\alpha.\, t \mid \mathsf{return}\ v \mid$
	$\mathsf{let}\ x = v(s); t \mid \mathsf{let}\ x : P = v(s); t$
Argument lists	$s ::= \epsilon \mid v, s$
Positive types	$P ::= \alpha \mid \downarrow N$
Negative types	$N ::= P \to N \mid \forall\alpha.\, N \mid \uparrow P$
Typing contexts	$\Theta ::= \cdot \mid \Theta, \alpha$
Typing environments	$\Gamma ::= \cdot \mid \Gamma, x : P$

Figure 1: Implicit Polarized F

$\boxed{\Theta \vdash A \text{ type}^{\pm}}$ In the context Θ, A is a well-formed positive/negative type

$$\frac{\alpha \in \Theta}{\Theta \vdash \alpha \text{ type}^{+}}\ \mathsf{Twfuvar} \qquad \frac{\Theta \vdash N \text{ type}^{-}}{\Theta \vdash \downarrow N \text{ type}^{+}}\ \mathsf{Twfshift}\!\downarrow \qquad \frac{\Theta, \alpha \vdash N \text{ type}^{-}}{\Theta \vdash \forall\alpha.\, N \text{ type}^{-}}\ \mathsf{Twfforall}$$

$$\frac{\Theta \vdash P \text{ type}^{+} \qquad \Theta \vdash N \text{ type}^{-}}{\Theta \vdash P \to N \text{ type}^{-}}\ \mathsf{Twfarrow} \qquad \frac{\Theta \vdash P \text{ type}^{+}}{\Theta \vdash \uparrow P \text{ type}^{-}}\ \mathsf{Twfshift}\!\uparrow$$

Figure 2: Well-formedness of declarative types

$\boxed{\Theta; \Gamma \vdash e : A}$ The term e synthesizes the type A

$\boxed{\Theta; \Gamma \vdash s : N \gg M}$ When passed to a head of type N, the argument list s synthesizes the type M

$$\frac{x : P \in \Gamma}{\Theta; \Gamma \vdash x : P}\ \text{Dvar} \qquad \frac{\Theta; \Gamma \vdash t : N}{\Theta; \Gamma \vdash \{t\} : \downarrow N}\ \text{Dthunk}$$

$$\frac{\Theta; \Gamma, x : P \vdash t : N}{\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N}\ \text{D}\lambda\text{abs} \qquad \frac{\Theta, \alpha; \Gamma \vdash t : N}{\Theta; \Gamma \vdash \Lambda \alpha.\, t : \forall \alpha.\, N}\ \text{Dgen}$$

$$\frac{\Theta; \Gamma \vdash v : P}{\Theta; \Gamma \vdash \text{return } v : \uparrow P}\ \text{Dreturn}$$

$$\frac{\Theta; \Gamma \vdash v : \downarrow M \qquad \Theta; \Gamma \vdash s : M \gg \uparrow Q \qquad \Theta \vdash \uparrow Q \leq^- \uparrow P \qquad \Theta; \Gamma, x : P \vdash t : N}{\Theta; \Gamma \vdash \text{let } x : P = v(s); t : N}\ \text{Dambiguouslet}$$

$$\frac{\Theta; \Gamma \vdash s : M \gg \uparrow Q \qquad \Theta; \Gamma, x : Q \vdash t : N \qquad \overset{\Theta; \Gamma \vdash v : \downarrow M}{\forall P.\ \text{if } \Theta; \Gamma \vdash s : M \gg \uparrow P \text{ then } \Theta \vdash Q \cong^+ P}}{\Theta; \Gamma \vdash \text{let } x = v(s); t : N}\ \text{Dunambiguouslet}$$

$$\frac{}{\Theta; \Gamma \vdash \epsilon : N \gg N}\ \text{Dspinenil} \qquad \frac{\Theta; \Gamma \vdash v : P \qquad \Theta \vdash P \leq^+ Q \qquad \Theta; \Gamma \vdash s : N \gg M}{\Theta; \Gamma \vdash v, s : (Q \to N) \gg M}\ \text{Dspinecons}$$

$$\frac{\Theta \vdash P \text{ type}^+ \qquad \Theta; \Gamma \vdash s : [P/\alpha]N \gg M}{\Theta; \Gamma \vdash s : (\forall \alpha.\, N) \gg M}\ \text{Dspinetypeabs}$$

Figure 3: Declarative type system

$\boxed{\Theta \vdash A \leq^{\pm} B}$ In the context Θ, A is a positive/negative declarative subtype of B

$$\frac{\Theta \vdash \alpha \text{ type}^+}{\Theta \vdash \alpha \leq^+ \alpha}\ {\leq^{\pm}}\text{Drefl} \qquad \frac{\Theta \vdash M \leq^- N \qquad \Theta \vdash N \leq^- M}{\Theta \vdash \downarrow N \leq^+ \downarrow M}\ {\leq^{\pm}}\text{Dshift}\downarrow \qquad \frac{\Theta, \alpha \vdash N \leq^- M}{\Theta \vdash N \leq^- \forall \alpha.\, M}\ {\leq^{\pm}}\text{Dforallr}$$

$$\frac{\Theta \vdash P \text{ type}^+ \qquad \Theta \vdash [P/\alpha]N \leq^- M}{\Theta \vdash \forall \alpha.\, N \leq^- M}\ {\leq^{\pm}}\text{Dforalll} \qquad \frac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash N \leq^- M}{\Theta \vdash P \to N \leq^- Q \to M}\ {\leq^{\pm}}\text{Darrow}$$

$$\frac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash P \leq^+ Q}{\Theta \vdash \uparrow P \leq^- \uparrow Q}\ {\leq^{\pm}}\text{Dshift}\uparrow$$

Figure 4: Declarative subtyping

$\boxed{\Theta \vdash A \cong^{\pm} B}$ In the context Θ, the types A and B are isomorphic

$$\Theta \vdash A \cong^{\pm} B \text{ iff } \Theta \vdash A \leq^{\pm} B \text{ and } \Theta \vdash B \leq^{\pm} A.$$

Figure 5: Isomorphic types

$\boxed{\Theta \vdash \Gamma_1 \cong \Gamma_2}$ In the context Θ, the environments Γ_1 and Γ_2 are isomorphic

$$\frac{}{\Theta \vdash \cdot \cong \cdot}\ \textsf{Eisoempty} \qquad \frac{\Theta \vdash \Gamma_1 \cong \Gamma_2 \qquad \Theta \vdash P \cong^{+} Q}{\Theta \vdash \Gamma_1, x : P \cong \Gamma_2, x : Q}\ \textsf{Eisovar}$$

Figure 6: Isomorphic environments

$\boxed{[\Theta]A}$ Applying a context Θ, as a substitution, to a type A

$$[\cdot]A = A \qquad [\Theta, \hat{\alpha}]A = [\Theta]A$$
$$[\Theta, \alpha]A = [\Theta]A \qquad [\Theta, \hat{\alpha} = P]A = [\Theta]([P/\hat{\alpha}]A)$$

Figure 7: Applying a context to a type

Positive types	$P ::= \ldots \mid \hat{\alpha}$
Contexts	$\Theta ::= \ldots \mid \Theta, \hat{\alpha} \mid \Theta, \hat{\alpha} = P$

Figure 8: Additions to declarative types and contexts to form their algorithmic counterparts

$\boxed{\Theta \vdash A \leq^{\pm} B \dashv \Theta'}$ In the context Θ, A checks algorithmically as a subtype of B, producing the context Θ'

$$\frac{}{\Theta_L, \alpha, \Theta_R \vdash \alpha \leq^{+} \alpha \dashv \Theta_L, \alpha, \Theta_R} \leq^{\pm}\mathsf{Arefl} \qquad \frac{\Theta_L \vdash P \text{ type}^{+} \qquad P \text{ ground}}{\Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^{+} \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R} \leq^{\pm}\mathsf{Ainst}$$

$$\frac{\Theta \vdash M \leq^{-} N \dashv \Theta' \qquad \Theta' \vdash N \leq^{-} [\Theta']M \dashv \Theta''}{\Theta \vdash {\downarrow}N \leq^{+} {\downarrow}M \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}{\downarrow}$$

$$\frac{\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^{-} M \dashv \Theta', \hat{\alpha}[= P] \qquad M \neq \forall \beta.\, M'}{\Theta \vdash \forall \alpha.\, N \leq^{-} M \dashv \Theta'} \leq^{\pm}\mathsf{Aforalll} \qquad \frac{\Theta, \alpha \vdash N \leq^{-} M \dashv \Theta', \alpha}{\Theta \vdash N \leq^{-} \forall \alpha.\, M \dashv \Theta'} \leq^{\pm}\mathsf{Aforallr}$$

$$\frac{\Theta \vdash Q \leq^{+} P \dashv \Theta' \qquad \Theta' \vdash [\Theta']N \leq^{-} M \dashv \Theta''}{\Theta \vdash P \to N \leq^{-} Q \to M \dashv \Theta''} \leq^{\pm}\mathsf{Aarrow}$$

$$\frac{\Theta \vdash Q \leq^{+} P \dashv \Theta' \qquad \Theta' \vdash [\Theta']P \leq^{+} Q \dashv \Theta''}{\Theta \vdash {\uparrow}P \leq^{-} {\uparrow}Q \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}{\uparrow}$$

Figure 9: Algorithmic subtyping

$\boxed{\Theta' \upharpoonright \Theta}$ Θ' restricted to only contain existential variables which appear in Θ

$$\frac{}{\cdot \upharpoonright \cdot = \cdot} \upharpoonright\mathsf{empty} \qquad \frac{\Theta' \upharpoonright \Theta = \Theta''}{\Theta', \alpha \upharpoonright \Theta, \alpha = \Theta'', \alpha} \upharpoonright\mathsf{uvar}$$

$$\frac{\Theta' \upharpoonright \Theta = \Theta''}{\Theta', \hat{\alpha}\,[= P] \upharpoonright \Theta, \hat{\alpha}\,[= Q] = \Theta'', \hat{\alpha}\,[= P]} \upharpoonright\mathsf{guessin} \qquad \frac{\Theta' \upharpoonright \Theta = \Theta'' \qquad \hat{\alpha}\,[= Q] \notin \Theta}{\Theta', \hat{\alpha}\,[= P] \upharpoonright \Theta = \Theta''} \upharpoonright\mathsf{guessnotin}$$

Figure 10: Definition of context restriction

$\boxed{\Theta; \Gamma \vdash e : A \dashv \Theta'}$ The term e synthesizes the type A, producing the context Θ'

$\boxed{\Theta; \Gamma \vdash s : N \gg M \dashv \Theta'}$ When passed to a head of type N, the argument list s synthesizes the type M, producing the context Θ'

$$\frac{x : P \in \Gamma}{\Theta; \Gamma \vdash x : P \dashv \Theta} \text{ Avar}$$

$$\frac{\Theta; \Gamma, x : P \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N \dashv \Theta'} \text{ A}\lambda\text{abs} \qquad \frac{\Theta, \alpha; \Gamma \vdash t : N \dashv \Theta', \alpha}{\Theta; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N \dashv \Theta'} \text{ Agen}$$

$$\frac{\Theta; \Gamma \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \{t\} : {\downarrow}N \dashv \Theta'} \text{ Athunk} \qquad \frac{\Theta; \Gamma \vdash v : P \dashv \Theta'}{\Theta; \Gamma \vdash \text{return } v : {\uparrow}P \dashv \Theta'} \text{ Areturn}$$

$$\frac{\begin{array}{c}\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta' \qquad \Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta'' \qquad \Theta'' \vdash P \leq^{+} Q \dashv \Theta''' \\ \Theta''' \vdash [\Theta''']Q \leq^{+} P \dashv \Theta^{(4)} \qquad \Theta^{(5)} = \Theta^{(4)} \upharpoonright \Theta \qquad \Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta^{(6)}\end{array}}{\Theta; \Gamma \vdash \text{let } x : P = v(s); t : N \dashv \Theta^{(6)}} \text{ Aambiguouslet}$$

$$\frac{\begin{array}{c}\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta' \\ \Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta'' \qquad \text{FEV}(Q) = \emptyset \qquad \Theta''' = \Theta'' \upharpoonright \Theta \qquad \Theta'''; \Gamma, x : Q \vdash t : N \dashv \Theta^{(4)}\end{array}}{\Theta; \Gamma \vdash \text{let } x = v(s); t : N \dashv \Theta^{(4)}} \text{ Aunambiguouslet}$$

$$\frac{}{\Theta; \Gamma \vdash \epsilon : N \gg N \dashv \Theta} \text{ Aspinenil}$$

$$\frac{\Theta; \Gamma \vdash v : P \dashv \Theta' \qquad \Theta' \vdash P \leq^{+} [\Theta']Q \dashv \Theta'' \qquad \Theta''; \Gamma \vdash s : [\Theta'']N \gg M \dashv \Theta'''}{\Theta; \Gamma \vdash v, s : Q \to N \gg M \dashv \Theta'''} \text{ Aspinecons}$$

$$\frac{\Theta; \Gamma \vdash s : N \gg M \dashv \Theta' \qquad \alpha \notin \text{FUV}(N)}{\Theta; \Gamma \vdash s : (\forall\alpha.\, N) \gg M \dashv \Theta'} \text{ Aspinetypeabsnotin}$$

$$\frac{\Theta, \hat{\alpha}; \Gamma \vdash s : [\hat{\alpha}/\alpha]N \gg M \dashv \Theta', \hat{\alpha}\,[= P] \qquad \alpha \in \text{FUV}(N)}{\Theta; \Gamma \vdash s : (\forall\alpha.\, N) \gg M \dashv \Theta', \hat{\alpha}\,[= P]} \text{ Aspinetypeabsin}$$

Figure 11: Algorithmic type system

$\boxed{\Theta\ \mathsf{ctx}}$ Θ is a well-formed context

$$\frac{}{\cdot\ \mathsf{ctx}}\ \textsf{Cwfempty} \qquad \frac{\Theta\ \mathsf{ctx}}{\Theta, \alpha\ \mathsf{ctx}}\ \textsf{Cwfuvar} \qquad \frac{\Theta\ \mathsf{ctx}}{\Theta, \hat{\alpha}\ \mathsf{ctx}}\ \textsf{Cwfunsolvedguess}$$

$$\frac{\Theta\ \mathsf{ctx} \qquad \Theta \vdash P\ \mathsf{type}^{+} \qquad P\ \mathsf{ground}}{\Theta, \hat{\alpha} = P\ \mathsf{ctx}}\ \textsf{Cwfsolvedguess}$$

Figure 12: Well-formedness of contexts

$\boxed{\Theta \vdash A\ \mathsf{type}^{\pm}}$ In the context Θ, A is a well-formed positive/negative type

$$\frac{\hat{\alpha} \in \mathrm{EV}(\Theta)}{\Theta \vdash \hat{\alpha}\ \mathsf{type}^{+}}\ \textsf{Twfguess}$$

Figure 13: Additional well-formedness rules for algorithmic types. $\mathrm{EV}(\Theta)$ contains all the existential type variables in Θ, independently of whether they are solved or unsolved.

$\boxed{\Theta \longrightarrow \Theta'}$ Θ extends to Θ'

$$\frac{}{\cdot \longrightarrow \cdot}\ \textsf{Cempty} \qquad \frac{\Theta \longrightarrow \Theta'}{\Theta, \alpha \longrightarrow \Theta', \alpha}\ \textsf{Cuvar} \qquad \frac{\Theta \longrightarrow \Theta'}{\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha}}\ \textsf{Cunsolvedguess}$$

$$\frac{\Theta \longrightarrow \Theta'}{\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} = P}\ \textsf{Csolveguess} \qquad \frac{\Theta \longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \longrightarrow \Theta', \hat{\alpha} = Q}\ \textsf{Csolvedguess}$$

Figure 14: Context extension

$\boxed{\Theta \Longrightarrow \Theta'}$ The context Θ' weakly extends the context Θ'

$$\frac{}{\cdot \Longrightarrow \cdot}\ \textsf{Wcempty} \qquad \frac{\Theta \Longrightarrow \Theta'}{\Theta, \alpha \Longrightarrow \Theta', \alpha}\ \textsf{Wcuvar} \qquad \frac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha}}\ \textsf{Wcunsolvedguess}$$

$$\frac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha} = P}\ \textsf{Wcsolveguess} \qquad \frac{\Theta \Longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \Longrightarrow \Theta', \hat{\alpha} = Q}\ \textsf{Wcsolvedguess}$$

$$\frac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha}}\ \textsf{Wcnewunsolvedguess} \qquad \frac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha} = P}\ \textsf{Wcnewsolvedguess}$$

Figure 15: Weak context extension. We highlight the rules that are “new” compared with context extension.

$\boxed{\Theta \vdash \Gamma\,\text{env}}$ The environment Γ is well-formed with respect to the context Θ

$$\frac{}{\Theta \vdash \cdot\ \text{env}}\ \textsf{Ewfempty} \qquad \frac{\Theta \vdash \Gamma\ \text{env} \qquad \Theta \vdash P\ \text{type}^{+} \qquad P\ \text{ground}}{\Theta \vdash \Gamma, x : P\ \text{env}}\ \textsf{Ewfvar}$$

Figure 16: Well-formedness of typing environments

$\boxed{|A|_{\text{NQ}}}$ The size of a type A, ignoring quantification

$$|\alpha|_{\text{NQ}} = 1 \qquad |{\downarrow}N|_{\text{NQ}} = |N|_{\text{NQ}} + 1 \qquad |P \to N|_{\text{NQ}} = |P|_{\text{NQ}} + |N|_{\text{NQ}} + 1$$
$$|\hat{\alpha}|_{\text{NQ}} = 1 \qquad |\forall\alpha.\,N|_{\text{NQ}} = |N|_{\text{NQ}} \qquad |{\uparrow}P|_{\text{NQ}} = |P|_{\text{NQ}} + 1$$

Figure 17: The size of a type, ignoring universal quantification

$\boxed{\text{NPQ}(A)}$ The number of prenex quantifiers in a type A

$$\text{NPQ}(\alpha) = 0 \qquad \text{NPQ}({\downarrow}N) = 0 \qquad \text{NPQ}(P \to N) = 0$$
$$\text{NPQ}(\hat{\alpha}) = 0 \qquad \text{NPQ}(\forall\alpha.\,N) = 1 + \text{NPQ}(N) \qquad \text{NPQ}({\uparrow}P) = 0$$

Figure 18: The number of prenex quantifiers in a type A

$\boxed{\|\Theta\|}$ The declarative context corresponding to the algorithmic context Θ

$$\|\cdot\| = \cdot \qquad \|\Theta, \alpha\| = \|\Theta\|, \alpha \qquad \|\Theta, \hat{\alpha}\| = \|\Theta\| \qquad \|\Theta, \hat{\alpha} = P\| = \|\Theta\|$$

Figure 19: Producing a declarative context from an algorithmic context

System F types

$$[\![A \to B]\!] = {\downarrow}([\![A]\!] \to {\uparrow}[\![B]\!])$$
$$[\![\forall\alpha.\,A]\!] = {\downarrow}\forall\alpha.\,{\uparrow}[\![A]\!]$$

System F terms

$$\begin{aligned}
[\![x]\!] &= \textsf{return}\ x \\
[\![\lambda x : A.\,e]\!] &= \textsf{return}\ \{\lambda x : [\![A]\!].\,[\![e]\!]\} \\
[\![e_1\ e_2]\!] &= \textsf{let}\ f[:\,P] = \{[\![e_1]\!]\}; \\
&\quad\ \textsf{let}\ x[:\,Q] = \{[\![e_2]\!]\}; \\
&\quad\ \textsf{let}\ y[:\,R] = f\ x; \\
&\quad\ \textsf{return}\ y \\
[\![\Lambda\alpha.\,e]\!] &= \Lambda\alpha.\,[\![e]\!] \\
[\![e\ [A]]\!] &= [\![e]\!]
\end{aligned}$$

Figure 20: An embedding of typeable terms in System F under a call-by-value evaluation order in Implicit Polarized F.

Lemmas

A Weakening

Lemma A.1 (Pushing uvars right preserves w.f.). *Let* $\Theta[\Theta_M]$ *abbreviate* $\Theta_L, \Theta_M, \Theta_R$. *Then if* $\Theta[\alpha, \Theta_M] \vdash A\ \text{type}^{\pm}$, $\Theta[\Theta_M, \alpha] \vdash A\ \text{type}^{\pm}$.

Lemma A.2 (Term well-formedness weakening). *If* $\Theta \vdash A\ \text{type}^{\pm}$ *then* $\Theta, \Theta' \vdash A\ \text{type}^{\pm}$.

Lemma A.3 (Pushing uvars right in declarative judgment). *Let* $\Theta[\Theta_M]$ *abbreviate* $\Theta_L, \Theta_M, \Theta_R$. *Then if* $\Theta[\alpha, \Theta_M] \vdash A \leq^{\pm} B$, $\Theta[\Theta_M, \alpha] \vdash A \leq^{\pm} B$.

Lemma A.4 (Declarative subtyping weakening). *If* $\Theta \vdash A \leq^{\pm} B$ *then* $\Theta, \Theta' \vdash A \leq^{\pm} B$.

B Declarative subtyping

Lemma B.1 (Declarative subtyping is reflexive). *If* $\Theta \vdash A\ \text{type}^{\pm}$ *then* $\Theta \vdash A \leq^{\pm} A$.

Lemma B.2 (Declarative substitution w.f.). *If* $\Theta_L, \Theta_R \vdash P\ \text{type}^{+}$ *and* $\Theta_L, \alpha, \Theta_R \vdash A\ \text{type}^{\pm}$, *then* $\Theta_L, \Theta_R \vdash [P/\alpha]A\ \text{type}^{\pm}$.

Lemma B.3 (Declarative subtyping is stable under substitution). *If* $\Theta_L, \Theta_R \vdash P\ \text{type}^{+}$, *then:*

- *If* $\Theta_L, \alpha, \Theta_R \vdash Q\ \text{type}^{+}$, $\Theta_L, \alpha, \Theta_R \vdash R\ \text{type}^{+}$, *and* $\Theta_L, \alpha, \Theta_R \vdash Q \leq^{+} R$, *then* $\Theta_L, \Theta_R \vdash [P/\alpha]Q \leq^{+} [P/\alpha]R$.
- *If* $\Theta_L, \alpha, \Theta_R \vdash N\ \text{type}^{-}$, $\Theta_L, \alpha, \Theta_R \vdash M\ \text{type}^{-}$, *and* $\Theta_L, \alpha, \Theta_R \vdash N \leq^{-} M$, *then* $\Theta_L, \Theta_R \vdash [P/\alpha]N \leq^{-} [P/\alpha]M$.

Lemma B.4 (Symmetry of positive declarative subtyping). *If* $\Theta \vdash P \leq^{+} Q$ *then* $\Theta \vdash Q \leq^{+} P$ *by a derivation of equal height.*

B.1 Isomorphic types

Lemma B.5 (Mutual subtyping substitution). *Given* $\Theta, \vec{\alpha} \vdash \vec{P}\ \text{type}^{+}$ *and* $\Theta, \vec{\beta} \vdash \vec{Q}\ \text{type}^{+}$*:*

- *If:*
  1. $\Theta, \vec{\alpha} \vdash R\ \text{type}^{+}$
  2. $\Theta, \vec{\beta} \vdash S\ \text{type}^{+}$
  3. $\Theta, \vec{\alpha} \vdash R \leq^{+} [\overrightarrow{P/\beta}]S$
  4. $\Theta, \vec{\beta} \vdash S \leq^{+} [\overrightarrow{Q/\alpha}]R$

  then:
  1. $\forall \beta_i \in \vec{\beta}.\ \beta_i \in \text{FUV}(S) \implies \exists \gamma.\ P_i = \gamma$
  2. $\forall \alpha_i \in \vec{\alpha}.\ \alpha_i \in \text{FUV}(R) \implies \exists \gamma.\ Q_i = \gamma$

- *If:*
  1. $\Theta, \vec{\alpha} \vdash M\ \text{type}^{-}$
  2. $\Theta, \vec{\beta} \vdash N\ \text{type}^{-}$
  3. $\Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}]N \leq^{-} M$
  4. $\Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]M \leq^{-} N$

  then:
  1. $\forall \beta_i \in \vec{\beta}.\ \beta_i \in \text{FUV}(N) \implies \exists \gamma.\ P_i = \gamma$
  2. $\forall \alpha_i \in \vec{\alpha}.\ \alpha_i \in \text{FUV}(M) \implies \exists \gamma.\ Q_i = \gamma$

Lemma B.6 (Isomorphic types are the same size). *If:*

1. $\Theta \vdash A\ \text{type}^{+}$
2. $\Theta \vdash B\ \text{type}^{+}$
3. $\Theta \vdash A \cong^{\pm} B$

then $|A|_{\text{NQ}} = |B|_{\text{NQ}}$.

B.2 Transitivity

Lemma B.7 (Declarative subtyping is transitive). *If* $\Theta \vdash A\ \text{type}^{\pm}$*,* $\Theta \vdash B\ \text{type}^{\pm}$*,* $\Theta \vdash C\ \text{type}^{\pm}$*,* $\Theta \vdash A \leq^{\pm} B$*, and* $\Theta \vdash B \leq^{\pm} C$*, then* $\Theta \vdash A \leq^{\pm} C$.

C Weak context extension

Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$). *If* $\Theta \longrightarrow \Theta'$*, then* $\Theta \Longrightarrow \Theta'$.

Lemma C.2 (Weak context extension is reflexive). *For all contexts* Θ*,* $\Theta \Longrightarrow \Theta$.

Lemma C.3 (Equality of declarative contexts (weak)). *If* $\Theta \Longrightarrow \Theta'$*, then* $\|\Theta\| = \|\Theta'\|$.

Lemma C.4 (Weak context extension is transitive). *If* $\Theta \Longrightarrow \Theta'$ *and* $\Theta' \Longrightarrow \Theta''$*, then* $\Theta \Longrightarrow \Theta''$.

Lemma C.5 (Weak context extension preserves well-formedness). *If* $\Theta \vdash A\ \text{type}^{\pm}$ *and* $\Theta \Longrightarrow \Theta'$ *then* $\Theta' \vdash A\ \text{type}^{\pm}$.

Lemma C.6 (Weak context extension preserves w.f. envs). *If* $\Theta \Longrightarrow \Theta'$ *and* $\Theta \vdash \Gamma\ \text{env}$*, then* $\Theta' \vdash \Gamma\ \text{env}$.

Lemma C.7 (The extended context makes the type ground (weak)). *If* $\Theta\ \text{ctx}$*,* $\Theta'\ \text{ctx}$*,* $\Theta \Longrightarrow \Theta'$*, and* $[\Theta'][\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

Lemma C.8 (Extending context preserves groundness (weak)). *If* $\Theta\ \text{ctx}$*,* $\Theta'\ \text{ctx}$*,* $\Theta \Longrightarrow \Theta'$*, and* $[\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

D Context extension

Lemma D.1 (Context extension is reflexive). *For all contexts* Θ*,* $\Theta \longrightarrow \Theta$.

Lemma D.2 (Equality of declarative contexts). *If* $\Theta \longrightarrow \Theta'$*, then* $\|\Theta\| = \|\Theta'\|$.

Lemma D.3 (Context extension is transitive). *If* $\Theta \longrightarrow \Theta'$ *and* $\Theta' \longrightarrow \Theta''$*, then* $\Theta \longrightarrow \Theta''$.

Lemma D.4 (Context extension preserves w.f.). *If* $\Theta \vdash A\ \text{type}^{\pm}$ *and* $\Theta \longrightarrow \Theta'$*, then* $\Theta' \vdash A\ \text{type}^{\pm}$.

Lemma D.5 (Applying a context to a ground type). *If* A *ground, then* $[\Theta]A = A$.

Lemma D.6 (Context application is idempotent). *If* $\Theta\ \text{ctx}$*, then* $[\Theta][\Theta]A = [\Theta]A$.

Lemma D.7 (The extended context makes the type ground). *If* $\Theta\ \text{ctx}$*,* $\Theta'\ \text{ctx}$*,* $\Theta \longrightarrow \Theta'$*, and* $[\Theta'][\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

Lemma D.8 (Extending context preserves groundness). *If* $\Theta\ \text{ctx}$*,* $\Theta'\ \text{ctx}$*,* $\Theta \longrightarrow \Theta'$*, and* $[\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

E Well-formedness of subtyping

Lemma E.1 (Applying context to the type preserves w.f.). *If* Θ ctx *and* $\Theta \vdash A$ type$^{\pm}$*, then* $\Theta \vdash [\Theta]A$ type$^{\pm}$.

Lemma E.2 (Algorithmic subtyping is w.f.).

- *If* $\Theta \vdash P \leq^{+} Q \dashv \Theta'$*,* Θ ctx*,* P *ground, and* $[\Theta]Q = Q$*, then* Θ' ctx*,* $\Theta \longrightarrow \Theta'$*, and* $[\Theta']Q$ *ground.*
- *If* $\Theta \vdash N \leq^{-} M \dashv \Theta'$*,* Θ ctx*,* M *ground, and* $[\Theta]N = N$*, then* Θ' ctx*,* $\Theta \longrightarrow \Theta'$*, and* $[\Theta']N$ *ground.*

F Soundness of subtyping

F.1 Lemmas for soundness

Lemma F.1 (Completing context preserves w.f.). *If* $\Theta \vdash A$ type$^{\pm}$ *and* A *ground then* $\|\Theta\| \vdash A$ type$^{\pm}$.

Lemma F.2 ($\Longrightarrow$ leads to isomorphic types). *If:*

1. $\Theta \vdash A$ type$^{\pm}$
2. $\Theta \Longrightarrow \Theta'$
3. $[\Theta']A$ *ground*
4. Θ ctx
5. Θ' ctx

then $\|\Theta\| \vdash [\Theta'][\Theta]A \cong^{\pm} [\Theta']A$.

Lemma F.3 ($\Longrightarrow$ leads to isomorphic types (ground)). *If:*

1. $\Theta \vdash A$ type$^{\pm}$
2. $[\Theta]A$ *ground*
3. $\Theta \Longrightarrow \Theta'$
4. Θ ctx
5. Θ' ctx

then $\|\Theta\| \vdash [\Theta]A \cong^{\pm} [\Theta']A$.

Lemma F.4 ($\longrightarrow$ leads to isomorphic types). *If:*

1. $\Theta \vdash A$ type$^{\pm}$
2. $\Theta \longrightarrow \Theta'$
3. $[\Theta']A$ *ground*
4. Θ ctx
5. Θ' ctx

then $\|\Theta\| \vdash [\Theta'][\Theta]A \cong^{\pm} [\Theta']A$.

Lemma F.5 ($\longrightarrow$ leads to isomorphic types (ground)). *If:*

1. $\Theta \vdash A \text{ type}^{\pm}$
2. $[\Theta]A$ *ground*
3. $\Theta \longrightarrow \Theta'$
4. $\Theta \text{ ctx}$
5. $\Theta' \text{ ctx}$

then $\|\Theta\| \vdash [\Theta]A \cong^{\pm} [\Theta']A$.

F.2 Statement

Theorem F.6 (Soundness of algorithmic subtyping). *Given a well-formed algorithmic context* Θ *and a well-formed complete context* Ω*:*

- *If* $\Theta \vdash P \leq^{+} Q \dashv \Theta'$, $\Theta' \longrightarrow \Omega$, P *ground,* $[\Theta]Q = Q$, $\Theta \vdash P \text{ type}^{+}$, *and* $\Theta \vdash Q \text{ type}^{+}$, *then* $\|\Theta\| \vdash P \leq^{+} [\Omega]Q$.
- *If* $\Theta \vdash N \leq^{-} M \dashv \Theta'$, $\Theta' \longrightarrow \Omega$, M *ground,* $[\Theta]N = N$, $\Theta \vdash N \text{ type}^{-}$, *and* $\Theta \vdash M \text{ type}^{-}$, *then* $\|\Theta\| \vdash [\Omega]N \leq^{-} M$.

G Completeness of subtyping

G.1 Lemmas for completeness

Lemma G.1 (Completion preserves w.f.). *If* $\Theta \text{ ctx}$, $\Theta \vdash A \text{ type}^{\pm}$, *and* $\Theta \longrightarrow \Omega$, *then* $\|\Theta\| \vdash [\Omega]A \text{ type}^{\pm}$.

Lemma G.2 (Extension solving guess). *If* $\Theta_L, \hat{\alpha}, \Theta_R \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R$ *and* $[\Omega_L]\Theta_L \vdash P \cong^{+} Q$, *then* $\Theta_L, \hat{\alpha} = P, \Theta_R \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R$.

Lemma G.3 (Context extension substitution size). *If:*

1. $\Theta \text{ ctx}$
2. $\Theta \vdash A \text{ type}^{\pm}$
3. $\Theta \longrightarrow \Omega$
4. $\Omega \text{ ctx}$

then $|[\Omega][\Theta]A|_{NQ} = |[\Omega]A|_{NQ}$.

Lemma G.4 (Context extension ground substitution size). *If:*

1. $\Theta \text{ ctx}$
2. $\Theta \vdash A \text{ type}^{\pm}$
3. $[\Theta]A$ *ground*
4. $\Theta \longrightarrow \Omega$
5. $\Omega \text{ ctx}$

then $|[\Theta]A|_{NQ} = |[\Omega]A|_{NQ}$.

G.2 Statement

Theorem G.5 (Completeness of algorithmic subtyping). *If* Θ ctx, $\Theta \longrightarrow \Omega$, *and* Ω ctx, *then:*

- *If* $\|\Theta\| \vdash P \le^+ [\Omega]Q$, $\Theta \vdash P\,\text{type}^+$, $\Theta \vdash Q\,\text{type}^+$, P *ground, and* $[\Theta]Q = Q$, *then* $\exists\Theta'$ *such that* $\Theta \vdash P \le^+ Q \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.
- *If* $\|\Theta\| \vdash [\Omega]N \le^- M$, $\Theta \vdash M\,\text{type}^-$, $\Theta \vdash N\,\text{type}^-$, M *ground, and* $[\Theta]N = N$, *then* $\exists\Theta'$ *such that* $\Theta \vdash N \le^- M \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.

H Determinism of subtyping

Lemma H.1 (Algorithmic subtyping is deterministic).

- *If* $\Theta \vdash P \le^+ Q \dashv \Theta'_1$ *and* $\Theta \vdash P \le^+ Q \dashv \Theta'_2$, *then* $\Theta'_1 = \Theta'_2$.
- *If* $\Theta \vdash N \le^- M \dashv \Theta'_1$ *and* $\Theta \vdash N \le^- M \dashv \Theta'_2$, *then* $\Theta'_1 = \Theta'_2$.

I Decidability of subtyping

I.1 Lemmas for decidability

Lemma I.1 (Completed non-ground size bounded by ground size).

- *If* $\Theta \vdash P \le^+ Q \dashv \Theta'$, Θ ctx, P *ground, and* $[\Theta]Q = Q$, *then* $|[\Theta']Q|_{NQ} \le |P|_{NQ}$.
- *If* $\Theta \vdash N \le^- M \dashv \Theta'$, Θ ctx, M *ground, and* $[\Theta]N = N$, *then* $|[\Theta']N|_{NQ} \le |M|_{NQ}$.

I.2 Statement

Lemma I.2 (Decidability of algorithmic subtyping). *There exists a total order* $\sqsubset$ *on well-formed algorithmic subtyping judgments such that for each derivation with subtyping judgment premises* A_i *and conclusion* B, *each* A_i *compares less than* B, *i.e.* $\forall i.\, A_i \sqsubset B$.

J Isomorphic types

Lemma J.1 (Isomorphic environments type the same terms). *If* $\Theta \vdash \Gamma \cong \Gamma'$, *then:*

- *If* $\Theta; \Gamma \vdash v : P$ *then* $\exists P'$ *such that* $\Theta \vdash P \cong^- P'$ *and* $\Theta; \Gamma' \vdash v : P'$.
- *If* $\Theta; \Gamma \vdash t : N$ *then* $\exists N'$ *such that* $\Theta \vdash N \cong^- N'$ *and* $\Theta; \Gamma' \vdash t : N'$.
- *If* $\Theta; \Gamma \vdash s : N \gg M$ *and* $\Theta \vdash N \cong^- N'$, *then* $\exists M'$ *such that* $\Theta \vdash M \cong^- M'$ *and* $\Theta; \Gamma \vdash s : N' \gg M'$.

K Well-formedness of typing

Lemma K.1 (Well-formedness of restricted contexts). *If* Θ ctx, Θ' ctx, $\Theta \Longrightarrow \Theta'$*, then* $\Theta' \restriction \Theta$ ctx, $\Theta \longrightarrow \Theta' \restriction \Theta$*, and* $\Theta' \restriction \Theta \Longrightarrow \Theta'$*.*

Lemma K.2 (Type well-formed with type variable removed). *If* $\Theta_L, \alpha, \Theta_R \vdash T\,\mathsf{type}^{\pm}$ *and* $\alpha \notin \mathrm{FUV}(T)$*, then* $\Theta_L, \Theta_R \vdash T\,\mathsf{type}^{\pm}$.

Lemma K.3 (Substitution preserves well-formedness of types). *If* $\Theta_L, \alpha, \Theta_R \vdash T\,\mathsf{type}^{\pm}$*, then* $\Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]T\,\mathsf{type}^{\pm}$.

Lemma K.4 (Context extension maintains variables). *If* $\Theta \longrightarrow \Omega$*, then* $\mathrm{FUV}(\Theta) = \mathrm{FUV}(\Omega)$ *and* $\mathrm{FEV}(\Theta) = \mathrm{FEV}(\Omega)$.

Lemma K.5 (Algorithmic typing is w.f.). *Given a typing context* Θ *and typing environment* Γ *such that* Θ ctx *and* $\Theta \vdash \Gamma$ env*:*

- *If* $\Theta; \Gamma \vdash v : P \dashv \Theta'$*, then* Θ' ctx*,* $\Theta \longrightarrow \Theta'$*,* $\Theta' \vdash P\,\mathsf{type}^{+}$*, and* P *ground.*
- *If* $\Theta; \Gamma \vdash t : N \dashv \Theta'$*, then* Θ' ctx*,* $\Theta \longrightarrow \Theta'$*,* $\Theta' \vdash N\,\mathsf{type}^{-}$*, and* N *ground.*
- *If* $\Theta; \Gamma \vdash s : N \gg M \dashv \Theta'$*,* $\Theta \vdash N\,\mathsf{type}^{-}$*, and* $[\Theta]N = N$*, then* Θ' ctx*,* $\Theta \Longrightarrow \Theta'$*,* $\Theta' \vdash M\,\mathsf{type}^{-}$*,* $[\Theta']M = M$*, and* $\mathrm{FEV}(M) \subseteq \mathrm{FEV}(N) \cup (\mathrm{FEV}(\Theta') \setminus \mathrm{FEV}(\Theta))$.

L Determinism of typing

Lemma L.1 (Algorithmic typing is deterministic).

- *If* $\Theta; \Gamma \vdash e : A_1 \dashv \Theta'_1$ *and* $\Theta; \Gamma \vdash e : A_2 \dashv \Theta'_2$*, then* $A_1 = A_2$ *and* $\Theta'_1 = \Theta'_2$*.*
- *If* $\Theta; \Gamma \vdash t : N \gg M_1 \dashv \Theta'_1$ *and* $\Theta; \Gamma \vdash t : N \gg M_2 \dashv \Theta'_2$*, then* $M_1 = M_2$ *and* $\Theta'_1 = \Theta'_2$*.*

M Decidability of typing

Lemma M.1 (Decidability of algorithmic typing). *There exists a total order* $\sqsubset$ *on well-formed algorithmic typing judgments such that for each derivation with typing judgment premises* A_i *and conclusion* B*, each* A_i *compares less than* B*, i.e.* $\forall i.\, A_i \sqsubset B$.

N Soundness of typing

N.1 Lemmas

Lemma N.1 (Extended complete context). *If* Θ' ctx*,* Ω ctx*,* $\Theta \longrightarrow \Omega$*,* $\Theta \Longrightarrow \Theta'$*, and* $\Theta' \restriction \Theta \longrightarrow \Omega$*, then* $\exists \Omega'$ *such that* Ω' ctx*,* $\Theta' \longrightarrow \Omega'$*, and* $\Omega \Longrightarrow \Omega'$*.*

Lemma N.2 (Identical restricted contexts). *If* Θ' ctx *and* $\Theta \longrightarrow \Theta'$*, then* $\Theta'' \restriction \Theta = \Theta'' \restriction \Theta'$*.*

N.2 Statement

Theorem N.3 (Soundness of algorithmic typing). *If* Θ ctx*,* $\Theta \vdash \Gamma$ env*,* $\Theta' \longrightarrow \Omega$*, and* Ω ctx*, then:*

- *If* $\Theta; \Gamma \vdash v : P \dashv \Theta'$*, then* $\|\Theta\|; \Gamma \vdash v : [\Omega]P$.
- *If* $\Theta; \Gamma \vdash t : N \dashv \Theta'$*, then* $\|\Theta\|; \Gamma \vdash t : [\Omega]N$.
- *If* $\Theta; \Gamma \vdash s : N \gg M \dashv \Theta'$*,* $\Theta \vdash N\ \text{type}^-$*, and* $[\Theta]N = N$*, then* $\exists M'$ *such that* $\|\Theta\| \vdash [\Omega]M \cong^- M'$ *and* $\|\Theta\|; \Gamma \vdash s : [\Omega]N \gg M'$.

O Completeness of typing

O.1 Lemmas

Lemma O.1 (Weak context extension maintains variables). *If* $\Theta \Longrightarrow \Theta'$ *then* $\mathrm{FEV}(\Theta) \subseteq \mathrm{FEV}(\Theta')$ *and* $\mathrm{FUV}(\Theta) = \mathrm{FUV}(\Theta')$.

Lemma O.2 (Reversing context extension from a complete context). *If* $\Omega \longrightarrow \Theta$ *then* $\Theta \longrightarrow \Omega$.

Lemma O.3 (Pulling back restricted contexts). *If* $\Theta \longrightarrow \Theta'$ *and* $\Theta' \upharpoonright \Theta'' \longrightarrow \Theta'''$*, then* $\Theta \upharpoonright \Theta'' \longrightarrow \Theta'''$.

O.2 Statement

Theorem O.4 (Completeness of algorithmic typing). *If* Θ ctx*,* $\Theta \vdash \Gamma$ env*,* $\Theta \longrightarrow \Omega$*, and* Ω ctx*, then:*

- *If* $\|\Theta\|; \Gamma \vdash v : P$ *then* $\exists \Theta'$ *such that* $\Theta; \Gamma \vdash v : P \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.
- *If* $\|\Theta\|; \Gamma \vdash t : N$ *then* $\exists \Theta'$ *such that* $\Theta; \Gamma \vdash t : N \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.
- *If* $\|\Theta\|; \Gamma \vdash s : [\Omega]N \gg M$*,* $\Theta \vdash N\ \text{type}^-$*, and* $[\Theta]N = N$*, then* $\exists \Theta', \Omega'$ *and* M' *such that* $\Theta; \Gamma \vdash s : N \gg M' \dashv \Theta'$*,* $\Omega \Longrightarrow \Omega'$*,* $\Theta' \longrightarrow \Omega'$*,* $\|\Theta\| \vdash [\Omega']M' \cong^- M$*,* $[\Theta']M' = M'$*, and* Ω' ctx.

Proofs

A' Weakening

Lemma A.1 (Pushing uvars right preserves w.f.). *Let* $\Theta[\Theta_M]$ *abbreviate* $\Theta_L, \Theta_M, \Theta_R$*. Then if* $\Theta[\alpha, \Theta_M] \vdash A\ \text{type}^{\pm}$*,* $\Theta[\Theta_M, \alpha] \vdash A\ \text{type}^{\pm}$.

Proof. By rule induction on $\Theta[\alpha, \Theta_M] \vdash A\ \text{type}^{\pm}$.

- **Case** $$\frac{\beta \in \mathrm{UV}(\Theta[\alpha, \Theta_M])}{\Theta[\alpha, \Theta_M] \vdash \beta\ \text{type}^+}\ \text{Twfuvar}$$

	$\beta \in \mathrm{UV}(\Theta[\alpha, \Theta_M])$	Subderivation
	$\beta \in \mathrm{UV}(\Theta[\Theta_M, \alpha])$	Since UV ignores order
☞	$\Theta[\Theta_M, \alpha] \vdash \beta\ \ \text{type}^+$	By Twfuvar

- **Case** $\dfrac{\hat{\alpha} \in EV(\Theta[\alpha, \Theta_M])}{\Theta[\alpha, \Theta_M] \vdash \hat{\alpha}\ \mathsf{type}^+}$ Twfguess

$$\begin{array}{lll}
 & \hat{\alpha} \in EV(\Theta[\alpha, \Theta_M]) & \text{Subderivation} \\
 & \hat{\alpha} \in EV(\Theta[\Theta_M, \alpha]) & \text{Since EV ignores order} \\
☞ & \Theta[\Theta_M, \alpha] \vdash \hat{\alpha}\ \ \mathsf{type}^+ & \text{By Twfguess}
\end{array}$$

- **Case** $\dfrac{\Theta[\alpha, \Theta_M] \vdash N\ \mathsf{type}^-}{\Theta[\alpha, \Theta_M] \vdash {\downarrow}N\ \mathsf{type}^+}$ Twfshift↓

$$\begin{array}{lll}
 & \Theta[\alpha, \Theta_M] \vdash N\ \ \mathsf{type}^- & \text{Subderivation} \\
 & \Theta[\Theta_M, \alpha] \vdash N\ \ \mathsf{type}^- & \text{By i.h.} \\
☞ & \Theta[\Theta_M, \alpha] \vdash {\downarrow}N\ \ \mathsf{type}^+ & \text{By Twfshift}\downarrow
\end{array}$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_M, \Theta_R, \beta \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_M, \Theta_R \vdash \forall\beta.\, N\ \mathsf{type}^-}$ Twfforall

$$\begin{array}{lll}
 & \Theta_L, \alpha, \Theta_M, \Theta_R, \beta \vdash N\ \ \mathsf{type}^- & \text{Subderivation} \\
 & \Theta_L, \Theta_M, \alpha, \Theta_R, \beta \vdash N\ \ \mathsf{type}^- & \text{By i.h.} \\
☞ & \Theta_L, \Theta_M, \alpha, \Theta_R \vdash \forall\beta.\, N\ \ \mathsf{type}^- & \text{By Twfforall}
\end{array}$$

- **Case** $\dfrac{\Theta[\alpha, \Theta_M] \vdash P\ \mathsf{type}^+ \qquad \Theta[\alpha, \Theta_M] \vdash N\ \mathsf{type}^-}{\Theta[\alpha, \Theta_M] \vdash P \to N\ \mathsf{type}^-}$ Twfarrow

$$\begin{array}{lll}
 & \Theta[\alpha, \Theta_M] \vdash P\ \ \mathsf{type}^+ & \text{Subderivation} \\
 & \Theta[\Theta_M, \alpha] \vdash P\ \ \mathsf{type}^+ & \text{By i.h.} \\
 & \Theta[\alpha, \Theta_M] \vdash N\ \ \mathsf{type}^- & \text{Subderivation} \\
 & \Theta[\Theta_M, \alpha] \vdash N\ \ \mathsf{type}^- & \text{By i.h.} \\
☞ & \Theta[\Theta_M, \alpha] \vdash P \to N\ \ \mathsf{type}^- & \text{By Twfarrow}
\end{array}$$

- **Case** $\dfrac{\Theta[\alpha, \Theta_M] \vdash P\ \mathsf{type}^+}{\Theta[\alpha, \Theta_M] \vdash {\uparrow}P\ \mathsf{type}^-}$ Twfshift↑

$$\begin{array}{lll}
 & \Theta[\alpha, \Theta_M] \vdash P\ \ \mathsf{type}^+ & \text{Subderivation} \\
 & \Theta[\Theta_M, \alpha] \vdash P\ \ \mathsf{type}^+ & \text{By i.h.} \\
☞ & \Theta[\Theta_M, \alpha] \vdash {\uparrow}P\ \ \mathsf{type}^- & \text{By Twfshift}\uparrow
\end{array}$$

□

Lemma A.2 (Term well-formedness weakening). *If* $\Theta \vdash A\ \text{type}^{\pm}$ *then* $\Theta, \Theta' \vdash A\ \text{type}^{\pm}$.

Proof. By rule induction on $\Theta \vdash A\ \text{type}^{\pm}$.

- **Case** $\dfrac{\alpha \in \mathrm{UV}(\Theta)}{\Theta \vdash \alpha\ \text{type}^{+}}\ \mathsf{Twfuvar}$

$$\begin{array}{rll} & \alpha \in \mathrm{UV}(\Theta) & \text{Subderivation} \\ & \alpha \in \mathrm{UV}(\Theta, \Theta') & \text{Since } \mathrm{UV}(\Theta) \subseteq \mathrm{UV}(\Theta, \Theta') \\ ☞ & \Theta, \Theta' \vdash \alpha\ \ \text{type}^{+} & \text{By } \mathsf{Twfuvar} \end{array}$$

- **Case** $\dfrac{\hat{\alpha} \in \mathrm{EV}(\Theta)}{\Theta \vdash \hat{\alpha}\ \text{type}^{+}}\ \mathsf{Twfguess}$

$$\begin{array}{rll} & \hat{\alpha} \in \mathrm{EV}(\Theta) & \text{Subderivation} \\ & \hat{\alpha} \in \mathrm{EV}(\Theta, \Theta') & \text{Since } \mathrm{EV}(\Theta) \subseteq \mathrm{EV}(\Theta, \Theta') \\ ☞ & \Theta, \Theta' \vdash \hat{\alpha}\ \ \text{type}^{+} & \text{By } \mathsf{Twfguess} \end{array}$$

- **Case** $\dfrac{\Theta \vdash N\ \text{type}^{-}}{\Theta \vdash {\downarrow}N\ \text{type}^{+}}\ \mathsf{Twfshift}{\downarrow}$

$$\begin{array}{rll} & \Theta \vdash N\ \ \text{type}^{-} & \text{Subderivation} \\ & \Theta, \Theta' \vdash N\ \ \text{type}^{-} & \text{By i.h.} \\ ☞ & \Theta, \Theta' \vdash {\downarrow}N\ \ \text{type}^{+} & \text{By } \mathsf{Twfshift}{\downarrow} \end{array}$$

- **Case** $\dfrac{\Theta, \alpha \vdash N\ \text{type}^{-}}{\Theta \vdash \forall \alpha.\, N\ \text{type}^{-}}\ \mathsf{Twfforall}$

$$\begin{array}{rll} & \Theta, \alpha \vdash N\ \ \text{type}^{-} & \text{Subderivation} \\ & \Theta, \alpha, \Theta' \vdash N\ \ \text{type}^{-} & \text{By i.h.} \\ & \Theta, \Theta', \alpha \vdash N\ \ \text{type}^{-} & \text{By Lemma A.1 (Pushing uvars right preserves w.f.)} \\ ☞ & \Theta, \Theta' \vdash \forall \alpha.\, N\ \ \text{type}^{-} & \text{By } \mathsf{Twfforall} \end{array}$$

- **Case** $\dfrac{\Theta \vdash P\ \text{type}^{+} \qquad \Theta \vdash N\ \text{type}^{-}}{\Theta \vdash P \to N\ \text{type}^{-}}\ \mathsf{Twfarrow}$

$$\begin{array}{rl} \Theta \vdash P\ \ \text{type}^{+} & \text{Subderivation} \\ \Theta, \Theta' \vdash P\ \ \text{type}^{+} & \text{By i.h.} \\ \Theta \vdash N\ \ \text{type}^{-} & \text{Subderivation} \\ \Theta, \Theta' \vdash N\ \ \text{type}^{-} & \text{By i.h.} \end{array}$$

☞ $\Theta, \Theta' \vdash P \to N \;\; \mathsf{type}^-$ By Twfarrow

- **Case** $\dfrac{\Theta \vdash P \; \mathsf{type}^+}{\Theta \vdash \uparrow P \; \mathsf{type}^-}$ Twfshift↑

	$\Theta \vdash P \;\; \mathsf{type}^+$	Subderivation
	$\Theta, \Theta' \vdash P \;\; \mathsf{type}^+$	By i.h.
☞	$\Theta, \Theta' \vdash \uparrow P \;\; \mathsf{type}^-$	By Twfshift↑

□

Lemma A.3 (Pushing uvars right in declarative judgment). *Let* $\Theta[\Theta_M]$ *abbreviate* $\Theta_L, \Theta_M, \Theta_R$. *Then if* $\Theta[\alpha, \Theta_M] \vdash A \leq^{\pm} B$*,* $\Theta[\Theta_M, \alpha] \vdash A \leq^{\pm} B$.

Proof. By rule induction on $\Theta[\alpha, \Theta_M] \vdash A \leq^{\pm} B$.

- **Case** $\dfrac{\Theta[\alpha, \Theta_M] \vdash \beta \; \mathsf{type}^+}{\Theta[\alpha, \Theta_M] \vdash \beta \leq^+ \beta} \leq^{\pm}$Drefl

	$\Theta[\alpha, \Theta_M] \vdash \beta \;\; \mathsf{type}^+$	Subderivation
	$\Theta[\Theta_M, \alpha] \vdash \beta \;\; \mathsf{type}^+$	By Lemma A.1 (Pushing uvars right preserves w.f.)
☞	$\Theta[\Theta_M, \alpha] \vdash \beta \leq^+ \beta$	By $\leq^{\pm}$Drefl

- **Case** $\dfrac{\Theta[\alpha, \Theta_M] \vdash M \leq^- N \qquad \Theta[\alpha, \Theta_M] \vdash N \leq^- M}{\Theta[\alpha, \Theta_M] \vdash \downarrow N \leq^+ \downarrow M} \leq^{\pm}$Dshift↓

	$\Theta[\alpha, \Theta_M] \vdash M \leq^- N$	Subderivation
	$\Theta[\Theta_M, \alpha] \vdash M \leq^- N$	By i.h.
	$\Theta[\alpha, \Theta_M] \vdash N \leq^- M$	Subderivation
	$\Theta[\Theta_M, \alpha] \vdash N \leq^- M$	By i.h.
☞	$\Theta[\Theta_M, \alpha] \vdash \downarrow N \leq^+ \downarrow M$	By $\leq^{\pm}$Dshift↓

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_M, \Theta_R, \beta \vdash N \leq^- M}{\Theta_L, \alpha, \Theta_M, \Theta_R \vdash N \leq^- \forall \beta. M} \leq^{\pm}$Dforallr

	$\Theta_L, \alpha, \Theta_M, \Theta_R, \beta \vdash N \leq^- M$	Subderivation
	$\Theta_L, \Theta_M, \alpha, \Theta_R, \beta \vdash N \leq^- M$	By i.h.
☞	$\Theta_L, \Theta_M, \alpha, \Theta_R \vdash N \leq^- \forall \beta. M$	By $\leq^{\pm}$Dforallr

- **Case** $\dfrac{\Theta[\alpha, \Theta_M] \vdash P\ \mathsf{type}^+ \qquad \Theta[\alpha, \Theta_M] \vdash [P/\beta]N \leq^- M}{\Theta[\alpha, \Theta_M] \vdash \forall\beta.\, N \leq^- M}\ \leq^{\pm}\mathsf{Dforalll}$

$$
\begin{array}{rll}
& \Theta[\alpha, \Theta_M] \vdash P\ \mathsf{type}^+ & \text{Subderivation} \\
& \Theta[\Theta_M, \alpha] \vdash P\ \mathsf{type}^+ & \text{By Lemma A.1 (Pushing uvars right preserves w.f.)} \\
& \Theta[\alpha, \Theta_M] \vdash [P/\beta]N \leq^- M & \text{Subderivation} \\
& \Theta[\Theta_M, \alpha] \vdash [P/\beta]N \leq^- M & \text{By i.h.} \\
☞ & \Theta[\Theta_M, \alpha] \vdash \forall\beta.\, N \leq^- M & \text{By } \leq^{\pm}\mathsf{Dforalll}
\end{array}
$$

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash N \leq^- M}{\Theta \vdash P \to N \leq^- Q \to M}\ \leq^{\pm}\mathsf{Darrow}$

$$
\begin{array}{rll}
& \Theta[\alpha, \Theta_M] \vdash Q \leq^+ P & \text{Subderivation} \\
& \Theta[\Theta_M, \alpha] \vdash Q \leq^+ P & \text{By i.h.} \\
& \Theta[\alpha, \Theta_M] \vdash N \leq^- M & \text{Subderivation} \\
& \Theta[\Theta_M, \alpha] \vdash N \leq^- M & \text{By i.h.} \\
☞ & \Theta[\Theta_M, \alpha] \vdash P \to N \leq^- Q \to M & \text{By } \leq^{\pm}\mathsf{Darrow}
\end{array}
$$

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash P \leq^+ Q}{\Theta \vdash {\uparrow}P \leq^- {\uparrow}Q}\ \leq^{\pm}\mathsf{Dshift}{\uparrow}$

$$
\begin{array}{rll}
& \Theta[\alpha, \Theta_M] \vdash Q \leq^+ P & \text{Subderivation} \\
& \Theta[\Theta_M, \alpha] \vdash Q \leq^+ P & \text{By i.h.} \\
& \Theta[\alpha, \Theta_M] \vdash P \leq^+ Q & \text{Subderivation} \\
& \Theta[\Theta_M, \alpha] \vdash P \leq^+ Q & \text{By i.h.} \\
☞ & \Theta[\Theta_M, \alpha] \vdash {\uparrow}P \leq^- {\uparrow}Q & \text{By } \leq^{\pm}\mathsf{Dshift}{\uparrow}
\end{array}
$$

□

Lemma A.4 (Declarative subtyping weakening). *If* $\Theta \vdash A \leq^{\pm} B$ *then* $\Theta, \Theta' \vdash A \leq^{\pm} B$.

Proof. By rule induction on $\Theta \vdash A \leq^{\pm} B$.

- **Case** $\dfrac{\Theta \vdash \alpha\ \mathsf{type}^+}{\Theta \vdash \alpha \leq^+ \alpha}\ \leq^{\pm}\mathsf{Drefl}$

$$
\begin{array}{rll}
& \Theta \vdash \alpha\ \mathsf{type}^+ & \text{Subderivation} \\
& \Theta, \Theta' \vdash \alpha\ \mathsf{type}^+ & \text{By Lemma A.2 (Term well-formedness weakening)} \\
☞ & \Theta, \Theta' \vdash \alpha \leq^+ \alpha & \text{By } \leq^{\pm}\mathsf{Drefl}
\end{array}
$$

- **Case** $\dfrac{\Theta \vdash M \leq^- N \qquad \Theta \vdash N \leq^- M}{\Theta \vdash {\downarrow}N \leq^+ {\downarrow}M} \leq^\pm\mathsf{Dshift}{\downarrow}$

$\Theta \vdash M \leq^- N$	Subderivation
$\Theta, \Theta' \vdash M \leq^- N$	By i.h.
$\Theta \vdash N \leq^- M$	Subderivation
$\Theta, \Theta' \vdash N \leq^- M$	By i.h.
☞ $\Theta, \Theta' \vdash {\downarrow}N \leq^+ {\downarrow}M$	By $\leq^\pm\mathsf{Dshift}{\downarrow}$

- **Case** $\dfrac{\Theta, \alpha \vdash N \leq^- M}{\Theta \vdash N \leq^- \forall\alpha.\, M} \leq^\pm\mathsf{Dforallr}$

$\Theta, \alpha \vdash N \leq^- M$	Subderivation
$\Theta, \alpha, \Theta' \vdash N \leq^- M$	By i.h.
$\Theta, \Theta', \alpha \vdash N \leq^- M$	By Lemma A.3 (Pushing uvars right in declarative judgment)
☞ $\Theta, \Theta' \vdash N \leq^- \forall\alpha.\, M$	By i.h.

- **Case** $\dfrac{\Theta \vdash P\ \mathsf{type}^+ \qquad \Theta \vdash [P/\alpha]N \leq^- M}{\Theta \vdash \forall\alpha.\, N \leq^- M} \leq^\pm\mathsf{Dforalll}$

$\Theta \vdash P\ \mathsf{type}^+$	Subderivation
$\Theta, \Theta' \vdash P\ \mathsf{type}^+$	By Lemma A.2 (Term well-formedness weakening)
$\Theta \vdash [P/\alpha]N \leq^- M$	Subderivation
$\Theta, \Theta' \vdash [P/\alpha]N \leq^- M$	By i.h.
☞ $\Theta, \Theta' \vdash \forall\alpha.\, N \leq^- M$	By $\leq^\pm\mathsf{Dforalll}$

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash N \leq^- M}{\Theta \vdash P \to N \leq^- Q \to M} \leq^\pm\mathsf{Darrow}$

$\Theta \vdash Q \leq^+ P$	Subderivation
$\Theta, \Theta' \vdash Q \leq^+ P$	By i.h.
$\Theta \vdash N \leq^- M$	Subderivation
$\Theta, \Theta' \vdash N \leq^- M$	By i.h.
☞ $\Theta, \Theta' \vdash P \to N \leq^- Q \to M$	By $\leq^\pm\mathsf{Darrow}$

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash P \leq^+ Q}{\Theta \vdash {\uparrow}P \leq^- {\uparrow}Q} \leq^\pm\mathsf{Dshift}{\uparrow}$

$$\begin{array}{rl}
\Theta \vdash Q \leq^+ P & \text{Subderivation} \\
\Theta, \Theta' \vdash Q \leq^+ P & \text{By i.h.} \\
\Theta \vdash P \leq^+ Q & \text{Subderivation} \\
\Theta, \Theta' \vdash P \leq^+ Q & \text{By i.h.} \\
☞ \quad \Theta, \Theta' \vdash \uparrow P \leq^- \uparrow Q & \text{By } \leq^{\pm}\text{Dshift}\uparrow
\end{array}$$

□

B' Declarative subtyping

Lemma B.1 (Declarative subtyping is reflexive). *If* $\Theta \vdash A \text{ type}^{\pm}$ *then* $\Theta \vdash A \leq^{\pm} A$.

Proof. By rule induction on $\Theta \vdash A \text{ type}^{\pm}$.

- **Case** $\dfrac{\alpha \in \text{UV}(\Theta)}{\Theta \vdash \alpha \text{ type}^+}$ Twfuvar

$$\begin{array}{rl}
\Theta \vdash \alpha \text{ type}^+ & \text{Assumption} \\
☞ \quad \Theta \vdash \alpha \leq^+ \alpha & \text{By } \leq^{\pm}\text{Drefl}
\end{array}$$

- **Case** $\dfrac{\Theta \vdash N \text{ type}^-}{\Theta \vdash \downarrow N \text{ type}^+}$ Twfshift$\downarrow$

$$\begin{array}{rl}
\Theta \vdash N \text{ type}^- & \text{Subderivation} \\
\Theta \vdash N \leq^- N & \text{By i.h.} \\
☞ \quad \Theta \vdash \downarrow N \leq^+ \downarrow N & \text{By } \leq^{\pm}\text{Dshift}\downarrow
\end{array}$$

- **Case** $\dfrac{\Theta, \alpha \vdash N \text{ type}^-}{\Theta \vdash \forall \alpha.\, N \text{ type}^-}$ Twfforall

$$\begin{array}{rl}
\Theta, \alpha \vdash N \text{ type}^- & \text{Subderivation} \\
\Theta, \alpha \vdash N \leq^- N & \text{By i.h.} \\
\alpha \in \text{UV}(\Theta, \alpha) & \text{By definition of UV} \\
\Theta, \alpha \vdash \alpha \text{ type}^+ & \text{By Twfuvar} \\
\Theta, \alpha \vdash \forall \alpha.\, N \leq^- N & \text{By } \leq^{\pm}\text{Dforalll} \\
☞ \quad \Theta \vdash \forall \alpha.\, N \leq^- \forall \alpha.\, N & \text{By } \leq^{\pm}\text{Dforallr}
\end{array}$$

- **Case** $\dfrac{\Theta \vdash P \text{ type}^+ \qquad \Theta \vdash N \text{ type}^-}{\Theta \vdash P \to N \text{ type}^-}$ Twfarrow

	$\Theta \vdash P\ \ \mathsf{type}^+$	Subderivation
	$\Theta \vdash P \leq^+ P$	By i.h.
	$\Theta \vdash N\ \ \mathsf{type}^-$	Subderivation
	$\Theta \vdash N \leq^- N$	By i.h.
☞	$\Theta \vdash P \to N \leq^- P \to N$	By $\leq^\pm$Darrow

- **Case** $\dfrac{\Theta \vdash P\ \mathsf{type}^+}{\Theta \vdash \uparrow P\ \mathsf{type}^-}$ Twfshift↑

	$\Theta \vdash P\ \ \mathsf{type}^+$	Subderivation
	$\Theta \vdash P \leq^+ P$	By i.h.
☞	$\Theta \vdash \uparrow P \leq^- \uparrow P$	By $\leq^\pm$Dshift↑

□

Lemma B.2 (Declarative substitution w.f.)**.** *If* $\Theta_L, \Theta_R \vdash P\ \mathsf{type}^+$ *and* $\Theta_L, \alpha, \Theta_R \vdash A\ \mathsf{type}^\pm$*, then* $\Theta_L, \Theta_R \vdash [P/\alpha]A\ \mathsf{type}^\pm$.

Proof. By rule induction on $\Theta_L, \alpha, \Theta_R \vdash A\ \mathsf{type}^\pm$.

- **Case** $\dfrac{\beta \in UV(\Theta_L, \alpha, \Theta_R)}{\Theta_L, \alpha, \Theta_R \vdash \beta\ \mathsf{type}^+}$ Twfuvar

Case $\beta = \alpha$:

	$[P/\alpha]\beta = P$	By definition of $[-]-$
☞	$\Theta_L, \Theta_R \vdash P\ \ \mathsf{type}^+$	Assumption

Case $\beta \neq \alpha$:

	$[P/\alpha]\beta = \beta$	By definition of $[-]-$
	$\beta \in UV(\Theta_L, \alpha, \Theta_R)$	Subderivation
	$\beta \in UV(\Theta_L, \Theta_R)$	Since $\beta \neq \alpha$
☞	$\Theta_L, \Theta_R \vdash \beta\ \ \mathsf{type}^+$	By Twfuvar

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_R \vdash \downarrow N\ \mathsf{type}^+}$ Twfshift↓

	$\Theta_L, \Theta_R \vdash P\ \ \mathsf{type}^+$	Assumption
	$\Theta_L, \alpha, \Theta_R \vdash N\ \ \mathsf{type}^-$	Subderivation
	$\Theta_L, \Theta_R \vdash [P/\alpha]N\ \ \mathsf{type}^-$	By i.h.
	$\Theta_L, \Theta_R \vdash \downarrow[P/\alpha]N\ \ \mathsf{type}^+$	By Twfshift↓
☞	$\Theta_L, \Theta_R \vdash [P/\alpha]\downarrow N\ \ \mathsf{type}^+$	By definition of $[-]-$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R, \beta \vdash N \text{ type}^-}{\Theta_L, \alpha, \Theta_R \vdash \forall\beta.\, N \text{ type}^-}$ Twfforall

$$\begin{array}{rll}
& \Theta_L, \Theta_R \vdash P \ \text{type}^+ & \text{Assumption} \\
& \Theta_L, \alpha, \Theta_R, \beta \vdash N \ \text{type}^- & \text{Subderivation} \\
& \Theta_L, \Theta_R, \beta \vdash P \ \text{type}^+ & \text{By Lemma A.2 (Term well-formedness weakening)} \\
& \Theta_L, \Theta_R, \beta \vdash [P/\alpha]N \ \text{type}^- & \text{By i.h.} \\
& \Theta_L, \Theta_R \vdash \forall\beta.\, [P/\alpha]N \ \text{type}^- & \text{By Twfforall} \\
☞ & \Theta_L, \Theta_R \vdash [P/\alpha]\forall\beta.\, N \ \text{type}^- & \text{By definition of } [-]-
\end{array}$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash Q \text{ type}^+ \qquad \Theta_L, \alpha, \Theta_R \vdash N \text{ type}^-}{\Theta_L, \alpha, \Theta_R \vdash Q \to N \text{ type}^-}$ Twfarrow

$$\begin{array}{rll}
& \Theta_L, \Theta_R \vdash P \ \text{type}^+ & \text{Assumption} \\
& \Theta_L, \alpha, \Theta_R \vdash Q \ \text{type}^+ & \text{Subderivation} \\
& \Theta_L, \Theta_R \vdash [P/\alpha]Q \ \text{type}^+ & \text{By i.h.} \\
& \Theta_L, \alpha, \Theta_R \vdash N \ \text{type}^- & \text{Subderivation} \\
& \Theta_L, \Theta_R \vdash [P/\alpha]N \ \text{type}^- & \text{By i.h.} \\
& \Theta_L, \Theta_R \vdash [P/\alpha]Q \to [P/\alpha]N \ \text{type}^- & \text{By Twfarrow} \\
☞ & \Theta_L, \Theta_R \vdash [P/\alpha](Q \to N) \ \text{type}^- & \text{By definition of } [-]-
\end{array}$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash Q \text{ type}^+}{\Theta_L, \alpha, \Theta_R \vdash \uparrow Q \text{ type}^-}$ Twfshift↑

$$\begin{array}{rll}
& \Theta_L, \Theta_R \vdash P \ \text{type}^+ & \text{Assumption} \\
& \Theta_L, \alpha, \Theta_R \vdash Q \ \text{type}^+ & \text{Subderivation} \\
& \Theta_L, \Theta_R \vdash [P/\alpha]Q \ \text{type}^+ & \text{By i.h.} \\
& \Theta_L, \Theta_R \vdash \uparrow[P/\alpha]Q \ \text{type}^- & \text{By Twfshift}\uparrow \\
☞ & \Theta_L, \Theta_R \vdash [P/\alpha]\uparrow Q \ \text{type}^- & \text{By definition of } [-]-
\end{array}$$

□

Lemma B.3 (Declarative subtyping is stable under substitution). *If* $\Theta_L, \Theta_R \vdash P \text{ type}^+$*, then:*

- *If* $\Theta_L, \alpha, \Theta_R \vdash Q \text{ type}^+$, $\Theta_L, \alpha, \Theta_R \vdash R \text{ type}^+$, *and* $\Theta_L, \alpha, \Theta_R \vdash Q \leq^+ R$, *then* $\Theta_L, \Theta_R \vdash [P/\alpha]Q \leq^+ [P/\alpha]R$.
- *If* $\Theta_L, \alpha, \Theta_R \vdash N \text{ type}^-$, $\Theta_L, \alpha, \Theta_R \vdash M \text{ type}^-$, *and* $\Theta_L, \alpha, \Theta_R \vdash N \leq^- M$, *then* $\Theta_L, \Theta_R \vdash [P/\alpha]N \leq^- [P/\alpha]M$.

Proof. By mutual rule induction on $\Theta_L, \alpha, \Theta_R \vdash Q \leq^+ R$ and $\Theta_L, \alpha, \Theta_R \vdash N \leq^- M$.

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash \beta \text{ type}^+}{\Theta_L, \alpha, \Theta_R \vdash \beta \leq^+ \beta}$ $\leq^\pm$Drefl

Case $\beta \neq \alpha$:

$$
\begin{array}{lrl}
 & [P/\alpha]\beta = \beta & \text{By definition of } [-]- \\
 & \Theta_L, \alpha, \Theta_R \vdash \beta \ \ \mathsf{type}^+ & \text{Subderivation} \\
 & \beta \in \mathrm{UV}(\Theta_L, \alpha, \Theta_R) & \text{Inversion (Twfuvar)} \\
 & \beta \in \mathrm{UV}(\Theta_L, \Theta_R) & \text{Since } \beta \neq \alpha \\
 & \Theta_L, \Theta_R \vdash \beta \ \ \mathsf{type}^+ & \text{By Twfuvar} \\
☞ & \Theta_L, \Theta_R \vdash \beta \leq^+ \beta & \text{By } \leq^{\pm}\mathsf{Drefl}
\end{array}
$$

Case $\beta = \alpha$:

$$
\begin{array}{lrl}
 & [P/\alpha]\beta = P & \text{By definition of } [-]- \\
 & \Theta_L, \Theta_R \vdash P \ \ \mathsf{type}^+ & \text{Assumption} \\
☞ & \Theta_L, \Theta_R \vdash P \leq^+ P & \text{By Lemma B.1 (Declarative subtyping is reflexive)}
\end{array}
$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash M \leq^- N \qquad \Theta_L, \alpha, \Theta_R \vdash N \leq^- M}{\Theta_L, \alpha, \Theta_R \vdash {\downarrow}N \leq^+ {\downarrow}M}\ \leq^{\pm}\mathsf{Dshift}{\downarrow}$

$$
\begin{array}{lrl}
 & \Theta_L, \Theta_R \vdash P \ \ \mathsf{type}^+ & \text{Assumption} \\
 & \Theta_L, \alpha, \Theta_R \vdash {\downarrow}N \ \ \mathsf{type}^+ & '' \\
 & \Theta_L, \alpha, \Theta_R \vdash N \ \ \mathsf{type}^- & \text{Inversion (Twfshift}{\downarrow}) \\
 & \Theta_L, \alpha, \Theta_R \vdash {\downarrow}M \ \ \mathsf{type}^+ & \text{Assumption} \\
 & \Theta_L, \alpha, \Theta_R \vdash M \ \ \mathsf{type}^- & \text{Inversion (Twfshift}{\downarrow}) \\
 & & \\
 & \Theta_L, \alpha, \Theta_R \vdash M \leq^- N & \text{Subderivation} \\
 & \Theta_L, \Theta_R \vdash [P/\alpha]M \leq^- [P/\alpha]N & \text{By i.h.} \\
 & \Theta_L, \alpha, \Theta_R \vdash M \leq^- N & \text{Subderivation} \\
 & \Theta_L, \Theta_R \vdash [P/\alpha]N \leq^- [P/\alpha]M & '' \\
☞ & \Theta_L, \Theta_R \vdash {\downarrow}[P/\alpha]N \leq^+ {\downarrow}[P/\alpha]M & \text{By } \leq^{\pm}\mathsf{Dshift}{\downarrow} \\
☞ & \Theta_L, \Theta_R \vdash [P/\alpha]{\downarrow}N \leq^+ [P/\alpha]{\downarrow}M & \text{By definition of } [-]-
\end{array}
$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R, \beta \vdash N \leq^- M}{\Theta_L, \alpha, \Theta_R \vdash N \leq^- \forall\beta.\, M}\ \leq^{\pm}\mathsf{Dforallr}$

$$
\begin{array}{lrl}
 & \Theta_L, \Theta_R \vdash P \ \ \mathsf{type}^+ & \text{Assumption} \\
 & \Theta_L, \alpha, \Theta_R \vdash N \ \ \mathsf{type}^- & '' \\
 & \Theta_L, \alpha, \Theta_R \vdash \forall\beta.\, M \ \ \mathsf{type}^- & '' \\
 & & \\
 & \Theta_L, \Theta_R, \beta \vdash P \ \ \mathsf{type}^+ & \text{By Lemma A.2 (Term well-formedness weakening)} \\
 & \Theta_L, \alpha, \Theta_R, \beta \vdash N \ \ \mathsf{type}^- & \text{By Lemma A.2 (Term well-formedness weakening)} \\
 & \Theta_L, \alpha, \Theta_R, \beta \vdash M \ \ \mathsf{type}^- & \text{Inversion (Twfforall)} \\
 & \Theta_L, \alpha, \Theta_R, \beta \vdash N \leq^- M & \text{Subderivation} \\
 & \Theta_L, \Theta_R, \beta \vdash [P/\alpha]N \leq^- [P/\alpha]M & \text{By i.h.} \\
☞ & \Theta_L, \Theta_R \vdash [P/\alpha]N \leq^- \forall\beta.\, [P/\alpha]M & \text{By } \leq^{\pm}\mathsf{Dforallr} \\
☞ & \Theta_L, \Theta_R \vdash [P/\alpha]N \leq^- [P/\alpha]\forall\beta.\, M & \text{By definition of } [-]-
\end{array}
$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash Q\ \mathsf{type}^+ \qquad \Theta_L, \alpha, \Theta_R \vdash [Q/\beta]N \leq^- M}{\Theta_L, \alpha, \Theta_R \vdash \forall\beta.\, N \leq^- M}\ \leq^{\pm}\mathsf{Dforalll}$

$\Theta_L, \Theta_R \vdash P\ \mathsf{type}^+$	Assumption
$\Theta_L, \alpha, \Theta_R \vdash \forall\beta.\, N\ \mathsf{type}^-$	"
$\Theta_L, \alpha, \Theta_R, \beta \vdash N\ \mathsf{type}^-$	Inversion (Twfforall)
$\Theta_L, \alpha, \Theta_R \vdash Q\ \mathsf{type}^+$	Subderivation
$\Theta_L, \alpha, \Theta_R \vdash [Q/\beta]N\ \mathsf{type}^-$	By Lemma B.2 (Declarative substitution w.f.)
$\Theta_L, \alpha, \Theta_R \vdash M\ \mathsf{type}^-$	Assumption
$\Theta_L, \alpha, \Theta_R \vdash [Q/\beta]N \leq^- M$	Subderivation
$\Theta_L, \Theta_R \vdash [P/\alpha][Q/\beta]N \leq^- [P/\alpha]M$	By i.h.
$\Theta_L, \Theta_R \vdash [([P/\alpha]Q)/\beta][P/\alpha]N \leq^- [P/\alpha]M$	Reordering substitutions
$\Theta_L, \Theta_R \vdash [P/\alpha]Q\ \mathsf{type}^+$	By Lemma B.2 (Declarative substitution w.f.)
☞ $\Theta_L, \Theta_R \vdash \forall\beta.\, [P/\alpha]N \leq^- [P/\alpha]M$	By $\leq^{\pm}$Dforalll (using $[P/\alpha]Q$ as the ground term)
☞ $\Theta_L, \Theta_R \vdash [P/\alpha]\forall\beta.\, N \leq^- [P/\alpha]M$	By definition of $[-]-$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash R \leq^+ Q \qquad \Theta_L, \alpha, \Theta_R \vdash N \leq^- M}{\Theta_L, \alpha, \Theta_R \vdash Q \to N \leq^- R \to M}\ \leq^{\pm}\mathsf{Darrow}$

$\Theta_L, \Theta_R \vdash P\ \mathsf{type}^-$	Assumption
$\Theta_L, \alpha, \Theta_R \vdash Q \to N\ \mathsf{type}^-$	"
$\Theta_L, \alpha, \Theta_R \vdash R \to M\ \mathsf{type}^-$	"
$\Theta_L, \alpha, \Theta_R \vdash R\ \mathsf{type}^+$	Inversion (Twfarrow)
$\Theta_L, \alpha, \Theta_R \vdash Q\ \mathsf{type}^+$	"
$\Theta_L, \alpha, \Theta_R \vdash R \leq^+ Q$	Subderivation
$\Theta_L, \Theta_R \vdash [P/\alpha]R \leq^+ [P/\alpha]Q$	By i.h.
$\Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-$	Inversion (Twfarrow)
$\Theta_L, \alpha, \Theta_R \vdash M\ \mathsf{type}^-$	"
$\Theta_L, \alpha, \Theta_R \vdash N \leq^- M$	Subderivation
$\Theta_L, \Theta_R \vdash [P/\alpha]N \leq^- [P/\alpha]M$	By i.h.
☞ $\Theta_L, \Theta_R \vdash [P/\alpha]Q \to [P/\alpha]N \leq^- [P/\alpha]R \to [P/\alpha]M$	By $\leq^{\pm}$Darrow
☞ $\Theta_L, \Theta_R \vdash [P/\alpha](Q \to N) \leq^- [P/\alpha](R \to M)$	By definition of $[-]-$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash R \leq^+ Q \qquad \Theta_L, \alpha, \Theta_R \vdash Q \leq^+ R}{\Theta_L, \alpha, \Theta_R \vdash \uparrow Q \leq^- \uparrow R}\ \leq^{\pm}\mathsf{Dshift}\uparrow$

$\Theta_L, \Theta_R \vdash P\ \mathsf{type}^+$	Assumption
$\Theta_L, \alpha, \Theta_R \vdash \uparrow Q\ \mathsf{type}^-$	"
$\Theta_L, \alpha, \Theta_R \vdash Q\ \mathsf{type}^+$	Inversion (Twfshift↑)
$\Theta_L, \alpha, \Theta_R \vdash \uparrow R\ \mathsf{type}^-$	Assumption
$\Theta_L, \alpha, \Theta_R \vdash R\ \mathsf{type}^+$	Inversion (Twfshift↑)

	$\Theta_L, \Theta_R \vdash [P/\alpha]R \le^+ [P/\alpha]Q$	By i.h.
	$\Theta_L, \Theta_R \vdash [P/\alpha]Q \le^+ [P/\alpha]R$	"
☞	$\Theta_L, \Theta_R \vdash \uparrow[P/\alpha]Q \le^- \uparrow[P/\alpha]R$	By $\le^\pm$Dshift$\uparrow$
☞	$\Theta_L, \Theta_R \vdash [P/\alpha]\uparrow Q \le^- [P/\alpha]\uparrow R$	By definition of $[-]-$

□

Lemma B.4 (Symmetry of positive declarative subtyping). *If* $\Theta \vdash P \le^+ Q$ *then* $\Theta \vdash Q \le^+ P$ *by a derivation of equal height.*

Proof. By rule induction on $\Theta \vdash P \le^+ Q$.

- **Case** $\dfrac{\Theta \vdash \alpha \text{ type}^+}{\Theta \vdash \alpha \le^+ \alpha} \le^\pm\text{Drefl}$

☞	$\Theta \vdash \alpha \le^+ \alpha$	Assumption

- **Case** $\dfrac{\Theta \vdash M \le^- N \qquad \Theta \vdash N \le^- M}{\Theta \vdash \downarrow N \le^+ \downarrow M} \le^\pm\text{Dshift}\downarrow$

	$\Theta \vdash N \le^- M$	Subderivation
	$\Theta \vdash M \le^- N$	"
☞	$\Theta \vdash \downarrow M \le^+ \downarrow N$	By $\le^\pm$Dshift$\downarrow$

□

B'.1 Isomorphic types

Lemma B.5 (Mutual subtyping substitution). *Given* $\Theta, \vec{\alpha} \vdash \vec{P}$ type$^+$ *and* $\Theta, \vec{\beta} \vdash \vec{Q}$ type$^+$*:*

- *If:*
  1. $\Theta, \vec{\alpha} \vdash R$ type$^+$
  2. $\Theta, \vec{\beta} \vdash S$ type$^+$
  3. $\Theta, \vec{\alpha} \vdash R \le^+ [\overrightarrow{P/\beta}]S$
  4. $\Theta, \vec{\beta} \vdash S \le^+ [\overrightarrow{Q/\alpha}]R$

  then:

  1. $\forall \beta_i \in \vec{\beta}.\, \beta_i \in \text{FUV}(S) \implies \exists \gamma.\, P_i = \gamma$
  2. $\forall \alpha_i \in \vec{\alpha}.\, \alpha_i \in \text{FUV}(R) \implies \exists \gamma.\, Q_i = \gamma$

- *If:*
  1. $\Theta, \vec{\alpha} \vdash M$ type$^-$
  2. $\Theta, \vec{\beta} \vdash N$ type$^-$
  3. $\Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}]N \le^- M$
  4. $\Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]M \le^- N$

  then:

  1. $\forall \beta_i \in \vec{\beta}.\, \beta_i \in \text{FUV}(N) \implies \exists \gamma.\, P_i = \gamma$
  2. $\forall \alpha_i \in \vec{\alpha}.\, \alpha_i \in \text{FUV}(M) \implies \exists \gamma.\, Q_i = \gamma$

Proof. By strong mutual rule induction on the pair of $\Theta, \vec{\alpha} \vdash R \le^+ [\overrightarrow{P/\beta}]S$ and $\Theta, \vec{\beta} \vdash S \le^+ [\overrightarrow{Q/\alpha}]R$, and the pair of $\Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}]N \le^- M$ and $\Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]M \le^- N$.

- **Case** $\dfrac{\Theta, \vec{\alpha} \vdash \gamma\ \text{type}^+}{\Theta, \vec{\alpha} \vdash \gamma \leq^+ \gamma} \leq^\pm\text{Drefl} \qquad \dfrac{\Theta, \vec{\beta} \vdash \gamma\ \text{type}^+}{\Theta, \vec{\beta} \vdash \gamma \leq^+ \gamma} \leq^\pm\text{Drefl}$

  By the $\leq^\pm$Drefl rule, we must have the same universal variable on both sides of both judgments.

	$[\overrightarrow{P/\beta}]S = \gamma$	Since we have an instance of $\leq^\pm$Drefl
☞	$P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in \text{FUV}(S)$
	$[\overrightarrow{Q/\alpha}]R = \gamma$	Since we have an instance of $\leq^\pm$Drefl
☞	$Q_i = \gamma$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in \text{FUV}(R)$

- **Case** $\dfrac{\Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}]N \leq^- M \qquad \Theta, \vec{\alpha} \vdash M \leq^- [\overrightarrow{P/\beta}]N}{\Theta, \vec{\alpha} \vdash {\downarrow}M \leq^+ [\overrightarrow{P/\beta}]{\downarrow}N} \leq^\pm\text{Dshift}{\downarrow}$

  If we have an instance of $\leq^\pm$Dshift↓, then the types on both sides of the other judgment must also start with ↓, so we must have another instance of $\leq^\pm$Dshift↓:

  $$\dfrac{\Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]M \leq^- N \qquad \Theta, \vec{\beta} \vdash N \leq^- [\overrightarrow{Q/\alpha}]M}{\Theta, \vec{\beta} \vdash {\downarrow}N \leq^+ [\overrightarrow{Q/\alpha}]{\downarrow}M} \leq^\pm\text{Dshift}{\downarrow}$$

	$\Theta, \vec{\alpha} \vdash M\ \text{type}^-$	Inversion (Twfshift↓)
	$\Theta, \vec{\beta} \vdash N\ \text{type}^-$	Inversion (Twfshift↓)
	$\Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}]N \leq^- M$	Subderivation
	$\Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]M \leq^- N$	″
	$P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in \text{FUV}(N)$ (by i.h.)
	$Q_i = \gamma$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in \text{FUV}(M)$ (by i.h.)
☞	$P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in \text{FUV}({\downarrow}N)$ (by definition of FUV)
☞	$Q_i = \gamma$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in \text{FUV}({\downarrow}M)$ (by definition of FUV)

- **Case** $\dfrac{\Theta, \vec{\alpha}, \gamma \vdash [\overrightarrow{P/\beta}]N \leq^- M}{\Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}]N \leq^- \forall\gamma.\, M} \leq^\pm\text{Dforallr}$

  By induction on the number of consecutive instances of $\leq^\pm$Dforallr in the derivation of the second judgment.

  - **Case** $\dfrac{\Theta, \vec{\beta} \vdash R\ \text{type}^+ \qquad \Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}, R/\gamma]M \leq^- N}{\Theta, \vec{\beta} \vdash \forall\gamma.\, [\overrightarrow{Q/\alpha}]M \leq^- N} \leq^\pm\text{Dforalll}$

    This is the base case of the inner induction. Our use of the outer induction hypothesis in this case is why we needed to perform a strong rule induction.

$\Theta, \vec{\alpha}, \gamma \vdash M\ \text{type}^-$	Inversion (Twfforall)
$\Theta, \vec{\beta} \vdash N\ \text{type}^-$	Assumption

$$\begin{array}{rll}
& \Theta, \vec{\alpha}, \gamma \vdash [\overrightarrow{P/\beta}]N \leq^- M & \text{Subderivation} \\
& \Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}, R/\gamma]M \leq^- N & '' \\
☞ & P_i = \delta & \text{For all } P_i \text{ such that } \beta_i \in \vec{\beta} \text{ and } \beta_i \in FUV(N) \text{ (by outer i.h.)} \\
& Q_i = \delta & \text{For all } Q_i \text{ such that } \alpha_i \in \vec{\alpha}, \gamma \text{ and } \alpha_i \in FUV(M) \text{ (by outer i.h.)} \\
☞ & Q_i = \delta & \text{For all } Q_i \text{ such that } \alpha_i \in \vec{\alpha} \text{ and } \alpha_i \in FUV(\forall\gamma.\, M) \\
& & \quad \text{(by definition of FUV)}
\end{array}$$

– **Case** $\dfrac{\Theta, \vec{\beta}, \delta \vdash [\overrightarrow{Q/\alpha}]\forall\gamma.\, M \leq^- N'}{\Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]\forall\gamma.\, M \leq^- \forall\delta.\, N'} \leq^{\pm}\text{Dforallr}$

This is the inductive step of the inner induction. Here we have $n = k + 1$ consecutive instances of $\leq^{\pm}$Dforallr in the derivation of the second judgment.

$$\begin{array}{rll}
& \Theta, \vec{\beta}, \delta \vdash N' \ \text{type}^- & \text{Inversion (Twfforall)} \\
& \Theta, \vec{\beta}, \delta \vdash [\overrightarrow{Q/\alpha}]\forall\gamma.\, M \leq^- N' & \text{Subderivation} \\
& \Theta, \delta, \vec{\beta} \vdash N' \ \text{type}^- & \text{By Lemma A.1 (Pushing uvars right preserves w.f.)} \\
& \Theta, \delta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]\forall\gamma.\, M \leq^- N' & \text{By Lemma A.3 (Pushing uvars right in declarative judgment)} \\
& P_i = \eta & \text{For all } P_i \text{ such that } \beta_i \in \vec{\beta} \text{ and } \beta_i \in FUV(N') \text{ (by inner i.h.)} \\
& Q_i = \eta & \text{For all } Q_i \text{ such that } \alpha_i \in \vec{\alpha}, \gamma \text{ and } \alpha_i \in FUV(M) \text{ (by inner i.h.)} \\
☞ & P_i = \eta & \text{For all } P_i \text{ such that } \beta_i \in \vec{\beta} \text{ and } \beta_i \in FUV(\forall\delta.\, N') \\
& & \quad \text{(by definition of FUV)} \\
☞ & Q_i = \eta & \text{For all } Q_i \text{ such that } \alpha_i \in \vec{\alpha} \text{ and } \alpha_i \in FUV(\forall\gamma.\, M) \\
& & \quad \text{(by definition of FUV)}
\end{array}$$

- **Case** $\dfrac{\Theta, \vec{\alpha} \vdash R \ \text{type}^+ \qquad \Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}, R/\gamma]N \leq^- M}{\Theta, \vec{\alpha} \vdash \forall\gamma.\, [\overrightarrow{P/\beta}]N \leq^- M} \leq^{\pm}\text{Dforalll}$

  We perform a case split over the derivation of the second judgment.

  – **Case** $\dfrac{\Theta, \vec{\beta} \vdash S \ \text{type}^+ \qquad \Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}, S/\delta]M' \leq^- \forall\gamma.\, N}{\Theta, \vec{\beta} \vdash \forall\delta.\, [\overrightarrow{Q/\alpha}]M' \leq^- \forall\gamma.\, N} \leq^{\pm}\text{Dforalll}$

$$\begin{array}{rll}
& \Theta, \vec{\alpha}, \delta \vdash M' \ \text{type}^- & \text{Inversion (Twfforall)} \\
& \Theta, \vec{\beta}, \gamma \vdash N \ \text{type}^- & '' \\
& \Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}, R/\gamma]N \leq^- \forall\delta.\, M' & \text{Subderivation} \\
& \Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}, S/\delta]M' \leq^- \forall\gamma.\, N & '' \\
& \Theta, \vec{\alpha}, \delta \vdash [\overrightarrow{P/\beta}, R/\gamma]N \leq^- M' & \text{Inversion } (\leq^{\pm}\text{Dforallr}) \\
& \Theta, \vec{\beta}, \gamma \vdash [\overrightarrow{Q/\alpha}, S/\delta]M' \leq^- N & '' \\
& P_i = \eta & \text{For all } P_i \text{ such that } \beta_i \in \vec{\beta}, \gamma \text{ and } \beta_i \in FUV(N) \text{ (by i.h.)} \\
& Q_i = \eta & \text{For all } Q_i \text{ such that } \alpha_i \in \vec{\alpha}, \delta \text{ and } \alpha_i \in FUV(M') \text{ (by i.h.)} \\
☞ & P_i = \eta & \text{For all } P_i \text{ such that } \beta_i \in \vec{\beta} \text{ and } \beta_i \in FUV(\forall\gamma.\, N) \\
& & \quad \text{(by definition of FUV)}
\end{array}$$

☞ $Q_i = \eta$ For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in FUV(\forall\delta.\,M')$ (by definition of FUV)

Here the application of the inductive hypothesis states that every universal variable in the arrays $\vec{\beta}, \gamma$ and $\vec{\alpha}, \delta$ that appears in the corresponding type is substituted by a universal variable (including γ and δ). As a result, the conclusion holds for just the universal variables in $\vec{\beta}$ and $\vec{\alpha}$.

– **Case** $\dfrac{\Theta, \vec{\beta}, \gamma \vdash \overrightarrow{[Q/\alpha]}M \leq^- N}{\Theta, \vec{\beta} \vdash \overrightarrow{[Q/\alpha]}M \leq^- \forall\gamma.\,N} \leq^{\pm}\text{Dforallr}$

$\Theta, \vec{\alpha} \vdash M\ \text{type}^-$	Assumption
$\Theta, \vec{\beta}, \gamma \vdash N\ \text{type}^-$	Inversion (Twfforall)
$\Theta, \vec{\alpha} \vdash \overrightarrow{[P/\beta, R/\gamma]}N \leq^- M$	Subderivation
$\Theta, \vec{\beta}, \gamma \vdash \overrightarrow{[Q/\alpha]}M \leq^- N$	$''$
$P_i = \delta$	For all P_i such that $\beta_i \in \vec{\beta}, \gamma$ and $\beta_i \in FUV(N)$ (by outer i.h.)
☞ $P_i = \delta$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in FUV(\forall\gamma.\,N)$ (by definition of FUV)
☞ $Q_i = \delta$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in FUV(M)$ (by outer i.h.)

- **Case** $\dfrac{\Theta, \vec{\alpha} \vdash R \leq^+ \overrightarrow{[P/\beta]}S \qquad \Theta, \vec{\alpha} \vdash \overrightarrow{[P/\beta]}N \leq^- M}{\Theta, \vec{\alpha} \vdash \overrightarrow{[P/\beta]}(S \to N) \leq^- R \to M} \leq^{\pm}\text{Darrow}$

If we have an instance of $\leq^{\pm}$Darrow, then the types on both sides of the other judgment must be function types, so we must have another instance of $\leq^{\pm}$Darrow:

$$\dfrac{\Theta, \vec{\beta} \vdash S \leq^+ \overrightarrow{[Q/\alpha]}R \qquad \Theta, \vec{\beta} \vdash \overrightarrow{[Q/\alpha]}M \leq^- N}{\Theta, \vec{\beta} \vdash \overrightarrow{[Q/\alpha]}(R \to M) \leq^- S \to N} \leq^{\pm}\text{Darrow}$$

$\Theta, \vec{\alpha} \vdash R\ \text{type}^+$	Inversion (Twfarrow)
$\Theta, \vec{\beta} \vdash S\ \text{type}^+$	Inversion (Twfarrow)
$\Theta, \vec{\alpha} \vdash R \leq^+ \overrightarrow{[P/\beta]}S$	Subderivation
$\Theta, \vec{\beta} \vdash S \leq^+ \overrightarrow{[Q/\alpha]}R$	$''$
$P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in FUV(S)$ (by i.h.)
$Q_i = \gamma$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in FUV(R)$ (by i.h.)
$\Theta, \vec{\alpha} \vdash M\ \text{type}^-$	Inversion (Twfarrow)
$\Theta, \vec{\beta} \vdash N\ \text{type}^-$	Inversion (Twfarrow)
$\Theta, \vec{\alpha} \vdash \overrightarrow{[P/\beta]}N \leq^- M$	Subderivation
$\Theta, \vec{\beta} \vdash \overrightarrow{[Q/\alpha]}M \leq^- N$	$''$
$P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in FUV(N)$ (by i.h.)
$Q_i = \gamma$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in FUV(M)$ (by i.h.)
☞ $P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in FUV(S \to N)$

(by definition of FUV)

☞ $Q_i = \gamma$ For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in \mathrm{FUV}(R \to M)$ (by definition of FUV)

- **Case** $$\dfrac{\Theta, \vec{\alpha} \vdash [\overrightarrow{P/\beta}]S \leq^{+} R \qquad \Theta, \vec{\alpha} \vdash R \leq^{+} [\overrightarrow{P/\beta}]S}{\Theta, \vec{\alpha} \vdash {\uparrow}R \leq^{-} [\overrightarrow{P/\beta}]{\uparrow}S} \leq^{\pm}\text{Dshift}{\uparrow}$$

  If we have an instance of $\leq^{\pm}$Dshift$\uparrow$, then the types on both sides of the other judgment must also start with $\uparrow$, so we must have another instance of $\leq^{\pm}$Dshift$\uparrow$:

$$\dfrac{\Theta, \vec{\beta} \vdash [\overrightarrow{Q/\alpha}]R \leq^{+} S \qquad \Theta, \vec{\beta} \vdash S \leq^{+} [\overrightarrow{Q/\alpha}]R}{\Theta, \vec{\beta} \vdash {\uparrow}S \leq^{-} [\overrightarrow{Q/\alpha}]{\uparrow}R} \leq^{\pm}\text{Dshift}{\uparrow}$$

	$\Theta, \vec{\alpha} \vdash R\ \text{type}^{+}$	Inversion (Twfshift$\uparrow$)
	$\Theta, \vec{\beta} \vdash S\ \text{type}^{+}$	Inversion (Twfshift$\uparrow$)
	$\Theta, \vec{\alpha} \vdash R \leq^{+} [\overrightarrow{P/\beta}]S$	Subderivation
	$\Theta, \vec{\beta} \vdash S \leq^{+} [\overrightarrow{Q/\alpha}]R$	″
	$P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in \mathrm{FUV}(S)$ (by i.h.)
	$Q_i = \gamma$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in \mathrm{FUV}(R)$ (by i.h.)
☞	$P_i = \gamma$	For all P_i such that $\beta_i \in \vec{\beta}$ and $\beta_i \in \mathrm{FUV}({\uparrow}S)$ (by definition of FUV)
☞	$Q_i = \gamma$	For all Q_i such that $\alpha_i \in \vec{\alpha}$ and $\alpha_i \in \mathrm{FUV}({\uparrow}R)$ (by definition of FUV)

□

Lemma B.6 (Isomorphic types are the same size). *If:*

1. $\Theta \vdash A\ \text{type}^{+}$
2. $\Theta \vdash B\ \text{type}^{+}$
3. $\Theta \vdash A \cong^{\pm} B$

then $|A|_{NQ} = |B|_{NQ}$.

Proof. By rule induction on $\Theta \vdash A \leq^{\pm} B$ and $\Theta \vdash B \leq^{\pm} A$.

- **Case** $$\dfrac{\Theta \vdash \gamma\ \text{type}^{+}}{\Theta \vdash \gamma \leq^{+} \gamma} \leq^{\pm}\text{Drefl} \qquad \dfrac{\Theta \vdash \gamma\ \text{type}^{+}}{\Theta \vdash \gamma \leq^{+} \gamma} \leq^{\pm}\text{Drefl}$$

☞ $|\gamma|_{NQ} = |\gamma|_{NQ}$ Identical LHS and RHS

- **Case** $\dfrac{\Theta \vdash N \leq^- M \qquad \Theta \vdash M \leq^- N}{\Theta \vdash {\downarrow}M \leq^+ {\downarrow}N} \leq^\pm\text{Dshift}{\downarrow} \qquad \dfrac{\Theta \vdash M \leq^- N \qquad \Theta \vdash N \leq^- M}{\Theta \vdash {\downarrow}N \leq^+ {\downarrow}M} \leq^\pm\text{Dshift}{\downarrow}$

	$\Theta \vdash {\downarrow}M\ \text{type}^+$	Assumption
	$\Theta \vdash M\ \text{type}^-$	Inversion (Twfshift↓)
	$\Theta \vdash {\downarrow}N\ \text{type}^+$	Assumption
	$\Theta \vdash N\ \text{type}^-$	Inversion (Twfshift↓)
	$\Theta \vdash M \leq^- N$	Subderivation
	$\Theta \vdash N \leq^- M$	Subderivation
	$\lvert M\rvert_{NQ} = \lvert N\rvert_{NQ}$	By i.h.
☞	$\lvert{\downarrow}M\rvert_{NQ} = \lvert{\downarrow}N\rvert_{NQ}$	By definition of $\lvert-\rvert_{NQ}$

- **Case** $\dfrac{\Theta \vdash P\ \text{type}^+ \qquad \Theta \vdash [P/\alpha]M \leq^- N}{\Theta \vdash \forall\alpha.\, M \leq^- N} \leq^\pm\text{Dforalll} \qquad \dfrac{\Theta, \alpha \vdash N \leq^- M}{\Theta \vdash N \leq^- \forall\alpha.\, M} \leq^\pm\text{Dforallr}$

	$\Theta \vdash P\ \text{type}^+$	Subderivation
	$\Theta \vdash N\ \text{type}^-$	Assumption
	$\Theta, \alpha \vdash N\ \text{type}^-$	By Lemma A.2 (Term well-formedness weakening)
	$\Theta \vdash \forall\alpha.\, M\ \text{type}^-$	Assumption
	$\Theta, \alpha \vdash M\ \text{type}^-$	Inversion (Twfforall)
	$\Theta \vdash [P/\alpha]M \leq^- N$	Subderivation
	$\Theta, \alpha \vdash N \leq^- M$	Subderivation
	$P \in \text{Uvar}$	By Lemma B.5 (Mutual subtyping substitution)
	$\lvert[P/\alpha]M\rvert_{NQ} = \lvert M\rvert_{NQ}$	Since $\lvert P\rvert_{NQ} = 1 = \lvert\alpha\rvert_{NQ}$
	$\Theta \vdash [P/\alpha]M\ \text{type}^-$	By Lemma B.2 (Declarative substitution w.f.)
	$\Theta \vdash N \leq^- [P/\alpha]M$	By Lemma B.3 (Declarative subtyping is stable under substitution)
	$\lvert[P/\alpha]M\rvert_{NQ} = \lvert N\rvert_{NQ}$	By i.h. (since $P \in \text{Uvar}$, the derivation of $\Theta \vdash N \leq^- [P/\alpha]M$ is the same size as the derivation of $\Theta, \alpha \vdash N \leq^- M$)
	$\lvert M\rvert_{NQ} = \lvert N\rvert_{NQ}$	Using $\lvert[P/\alpha]M\rvert_{NQ} = \lvert M\rvert_{NQ}$
☞	$\lvert\forall\alpha.\, M\rvert_{NQ} = \lvert N\rvert_{NQ}$	By definition of type size

- **Case** $\dfrac{\Theta, \alpha \vdash M \leq^- N}{\Theta \vdash M \leq^- \forall\alpha.\, N} \leq^\pm\text{Dforallr} \qquad \dfrac{\Theta \vdash P\ \text{type}^+ \Theta \vdash [P/\alpha]N \leq^- M}{\Theta \vdash \forall\alpha.\, N \leq^- M} \leq^\pm\text{Dforalll}$

  Symmetrical to the previous case (M and N are swapped).

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash M \leq^- N}{\Theta \vdash P \to M \leq^- Q \to N} \leq^\pm\text{Darrow} \qquad \dfrac{\Theta \vdash P \leq^+ Q \qquad \Theta \vdash N \leq^- M}{\Theta \vdash Q \to N \leq^- P \to M} \leq^\pm\text{Darrow}$

$\Theta \vdash P \to N\ \text{type}^-$	Assumption
$\Theta \vdash P\ \text{type}^+$	Inversion (Twfarrow)
$\Theta \vdash N\ \text{type}^-$	"
$\Theta \vdash Q \to M\ \text{type}^-$	Assumption

$\Theta \vdash Q\ \mathsf{type}^+$	Inversion (Twfarrow)
$\Theta \vdash M\ \mathsf{type}^-$	"
$\Theta \vdash P \le^+ Q$	Subderivation
$\Theta \vdash Q \le^+ P$	"
$\lvert P\rvert_{NQ} = \lvert Q\rvert_{NQ}$	By i.h.
$\Theta \vdash M \le^- N$	Subderivation
$\Theta \vdash N \le^- M$	"
$\lvert M\rvert_{NQ} = \lvert N\rvert_{NQ}$	By i.h.
☞ $\lvert P \to M\rvert_{NQ} = \lvert Q \to N\rvert_{NQ}$	By definition of type size

- **Case** $\dfrac{\Theta \vdash Q \le^+ P \qquad \Theta \vdash P \le^+ Q}{\Theta \vdash \uparrow P \le^- \uparrow Q} \le^\pm\mathsf{Dshift}\uparrow \qquad \dfrac{\Theta \vdash Q \le^+ P \qquad \Theta \vdash Q \le^+ P}{\Theta \vdash \uparrow Q \le^- \uparrow P} \le^\pm\mathsf{Dshift}\uparrow$

$\Theta \vdash \uparrow P\ \mathsf{type}^-$	Assumption
$\Theta \vdash P\ \mathsf{type}^+$	Inversion (Twfshift↑)
$\Theta \vdash \uparrow Q\ \mathsf{type}^-$	Assumption
$\Theta \vdash Q\ \mathsf{type}^+$	Inversion (Twfshift↑)
$\Theta \vdash P \le^+ Q$	Subderivation
$\Theta \vdash Q \le^+ P$	"
$\lvert P\rvert_{NQ} = \lvert Q\rvert_{NQ}$	By i.h.
$\lvert \uparrow P\rvert_{NQ} = \lvert \uparrow Q\rvert_{NQ}$	By definition of $\lvert - \rvert_{NQ}$

□

B'.2 Transitivity

Lemma B.7 (Declarative subtyping is transitive). *If* $\Theta \vdash A\ \mathsf{type}^\pm$, $\Theta \vdash B\ \mathsf{type}^\pm$, $\Theta \vdash C\ \mathsf{type}^\pm$, $\Theta \vdash A \le^\pm B$, *and* $\Theta \vdash B \le^\pm C$, *then* $\Theta \vdash A \le^\pm C$.

Proof. By rule induction on $\Theta \vdash B \le^\pm C$ weighted by the lexicographic ordering of $(\lvert B\rvert_{NQ}, \mathrm{NPQ}(B)+\mathrm{NPQ}(C))$ in the positive case and $(\lvert C\rvert_{NQ}, \mathrm{NPQ}(B)+\mathrm{NPQ}(C))$ in the negative case.

- **Case** $\dfrac{\Theta \vdash \alpha\ \mathsf{type}^+}{\Theta \vdash \alpha \le^+ \alpha} \le^\pm\mathsf{Drefl} \qquad \dfrac{\Theta \vdash \alpha\ \mathsf{type}^+}{\Theta \vdash \alpha \le^+ \alpha} \le^\pm\mathsf{Drefl}$

$\Theta \vdash \alpha\ \mathsf{type}^+$	Subderivation
☞ $\Theta \vdash \alpha \le^+ \alpha$	By $\le^\pm\mathsf{Drefl}$

- **Case** $\dfrac{\Theta \vdash M \le^- N \qquad \Theta \vdash N \le^- M}{\Theta \vdash \downarrow N \le^+ \downarrow M} \le^\pm\mathsf{Dshift}\downarrow \qquad \dfrac{\Theta \vdash N' \le^- M \qquad \Theta \vdash M \le^- N'}{\Theta \vdash \downarrow M \le^+ \downarrow N'} \le^\pm\mathsf{Dshift}\downarrow$

  The second judgment must be an instance of $\le^\pm\mathsf{Dshift}\downarrow$ due to the structure of $\downarrow M$.

$$
\begin{array}{rl}
\Theta \vdash {\downarrow}N \ \mathsf{type}^+ & \text{Assumption} \\
\Theta \vdash {\downarrow}M \ \mathsf{type}^+ & '' \\
\Theta \vdash {\downarrow}N' \ \mathsf{type}^+ & '' \\
\Theta \vdash N \ \mathsf{type}^- & \text{Inversion } (\mathsf{Twfshift}{\downarrow}) \\
\Theta \vdash M \ \mathsf{type}^- & '' \\
\Theta \vdash N' \ \mathsf{type}^- & ''
\end{array}
$$

$$
\begin{array}{rl}
\Theta \vdash M \leq^- N & \text{Subderivation} \\
\Theta \vdash N \leq^- M & '' \\
|M|_{\mathsf{NQ}} = |N|_{\mathsf{NQ}} & \text{By Lemma B.6 (Isomorphic types are the same size)} \\
\Theta \vdash M \leq^- N' & \text{Subderivation} \\
\Theta \vdash N' \leq^- M & '' \\
|M|_{\mathsf{NQ}} = |N'|_{\mathsf{NQ}} & \text{By Lemma B.6 (Isomorphic types are the same size)}
\end{array}
$$

$$
\begin{array}{rl}
\Theta \vdash N' \leq^- M & \text{Above} \\
\Theta \vdash M \leq^- N & '' \\
\Theta \vdash N' \leq^- N & \text{By i.h. } (|N|_{\mathsf{NQ}} = |M|_{\mathsf{NQ}} < |{\downarrow}M|_{\mathsf{NQ}}) \\
\Theta \vdash N \leq^- M & \text{Above} \\
\Theta \vdash M \leq^- N' & '' \\
\Theta \vdash N \leq^- N' & \text{By i.h. } (|N'|_{\mathsf{NQ}} = |M|_{\mathsf{NQ}} < |{\downarrow}M|_{\mathsf{NQ}}) \\
☞ \quad \Theta \vdash {\downarrow}N \leq^+ {\downarrow}N' & \text{By } \leq^\pm\mathsf{Dshift}{\downarrow}
\end{array}
$$

- **Case** $\dfrac{\Theta, \alpha \vdash M \leq^- N'}{\Theta \vdash M \leq^- \forall\alpha.\, N'} \leq^\pm\mathsf{Dforallr}$

  Here we only need to decompose the second declarative judgment.

$$
\begin{array}{rl}
\Theta \vdash N \ \mathsf{type}^- & \text{Assumption} \\
\Theta, \alpha \vdash N \ \mathsf{type}^- & \text{By Lemma A.2 (Term well-formedness weakening)} \\
\Theta \vdash M \ \mathsf{type}^- & \text{Assumption} \\
\Theta, \alpha \vdash M \ \mathsf{type}^- & \text{By Lemma A.2 (Term well-formedness weakening)} \\
\Theta, \alpha \vdash \forall\alpha.\, N' \ \mathsf{type}^- & \text{Assumption} \\
\Theta, \alpha \vdash N' \ \mathsf{type}^- & \text{Inversion } (\mathsf{Twfforall})
\end{array}
$$

$$
\begin{array}{rl}
\Theta \vdash N \leq^- M & \text{Assumption} \\
\Theta, \alpha \vdash N \leq^- M & \text{By Lemma A.4 (Declarative subtyping weakening)} \\
\Theta, \alpha \vdash M \leq^- N' & \text{Subderivation} \\
\Theta, \alpha \vdash N \leq^- N' & \text{By i.h. } (|N'|_{\mathsf{NQ}} = |\forall\alpha.\, N'|_{\mathsf{NQ}} \text{ and the number of prenex quantifiers in} \\
& \quad \text{the second judgment has reduced by 1)} \\
☞ \quad \Theta \vdash N \leq^- \forall\alpha.\, N' & \text{By } \leq^\pm\mathsf{Dforallr} \ (\alpha \notin \mathsf{FUV}(N) \text{ since } \alpha \notin \mathsf{UV}(\Theta) \text{ (because } \alpha \text{ fresh)} \\
& \quad \text{and also } \Theta \vdash N\,\mathsf{type}^-)
\end{array}
$$

- **Case** $\dfrac{\Theta, \alpha \vdash N \leq^- M}{\Theta \vdash N \leq^- \forall\alpha.\, M} \leq^\pm\mathsf{Dforallr} \qquad \dfrac{\Theta \vdash P\,\mathsf{type}^+ \qquad \Theta \vdash [P/\alpha]M \leq^- N'}{\Theta \vdash \forall\alpha.\, M \leq^- N'} \leq^\pm\mathsf{Dforalll}$

	$\Theta \vdash N \ \mathsf{type}^-$	Assumption
	$\Theta \vdash \forall\alpha.\, M \ \mathsf{type}^-$	Assumption
	$\Theta, \alpha \vdash M \ \mathsf{type}^-$	Inversion (Twfforall)
	$\Theta \vdash N' \ \mathsf{type}^-$	Assumption
	$\Theta, \alpha \vdash N \leq^- M$	Subderivation
	$\Theta \vdash P \ \mathsf{type}^+$	Subderivation
	$\Theta \vdash N \leq^- [P/\alpha]M$	By Lemma B.3 (Declarative subtyping is stable under substitution) ($\alpha \notin FUV(N)$ by side condition of $\leq^\pm$Dforallr)
	$\Theta \vdash [P/\alpha]M \leq^- N'$	Subderivation
☞	$\Theta \vdash N \leq^- N'$	By i.h. ($\lvert N'\rvert_{NQ} = \lvert N'\rvert_{NQ}$ and the number of prenex quantifiers in the second judgment has reduced by 1. Substitution can only replace positive types, so it cannot change the number of prenex quantifiers in a negative type)

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash N \leq^- M}{\Theta \vdash P \to N \leq^- Q \to M}\ \leq^\pm\mathsf{Darrow} \qquad \dfrac{\Theta \vdash P' \leq^+ Q \qquad \Theta \vdash M \leq^- N'}{\Theta \vdash Q \to M \leq^- P' \to N'}\ \leq^\pm\mathsf{Darrow}$

	$\Theta \vdash P \to N \ \mathsf{type}^-$	Assumption
	$\Theta \vdash P \ \mathsf{type}^+$	Inversion (Twfarrow)
	$\Theta \vdash N \ \mathsf{type}^-$	"
	$\Theta \vdash Q \to M \ \mathsf{type}^-$	Assumption
	$\Theta \vdash Q \ \mathsf{type}^+$	Inversion (Twfarrow)
	$\Theta \vdash M \ \mathsf{type}^-$	"
	$\Theta \vdash P' \to N' \ \mathsf{type}^-$	Assumption
	$\Theta \vdash P' \ \mathsf{type}^+$	Inversion (Twfarrow)
	$\Theta \vdash N' \ \mathsf{type}^-$	"
	$\Theta \vdash P' \leq^+ Q$	Subderivation
	$\Theta \vdash Q \leq^+ P'$	By Lemma B.4 (Symmetry of positive declarative subtyping)
	$\lvert P'\rvert_{NQ} = \lvert Q\rvert_{NQ}$	By Lemma B.6 (Isomorphic types are the same size)
	$\Theta \vdash P' \leq^+ Q$	Subderivation
	$\Theta \vdash Q \leq^+ P$	"
	$\Theta \vdash P' \leq^+ P$	By i.h. ($\lvert Q\rvert_{NQ} = \lvert P'\rvert_{NQ} < \lvert P' \to N'\rvert_{NQ}$)
	$\Theta \vdash N \leq^- M$	Subderivation
	$\Theta \vdash M \leq^- N'$	"
	$\Theta \vdash N \leq^- N'$	By i.h. ($\lvert N'\rvert_{NQ} < \lvert P' \to N'\rvert_{NQ}$)
☞	$\Theta \vdash P \to N \leq^- P' \to N'$	By $\leq^\pm$Darrow

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \qquad \Theta \vdash P \leq^+ Q}{\Theta \vdash {\uparrow}P \leq^- {\uparrow}Q}\ \leq^\pm\mathsf{Dshift}{\uparrow} \qquad \dfrac{\Theta \vdash P' \leq^+ Q \qquad \Theta \vdash Q \leq^+ P'}{\Theta \vdash {\uparrow}Q \leq^- {\uparrow}P'}\ \leq^\pm\mathsf{Dshift}{\uparrow}$

  Symmetrical to $\leq^\pm$Dshift$\downarrow$ case.

□

C' Weak context extension

Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)**.** *If* $\Theta \longrightarrow \Theta'$*, then* $\Theta \Longrightarrow \Theta'$*.*

Proof. By rule induction over the $\Theta \longrightarrow \Theta'$ judgment.

- **Case**
$$\frac{}{\cdot \longrightarrow \cdot}\ \textsf{Cempty}$$

☞ $\cdot \Longrightarrow \cdot$ By Wcempty

- **Case**
$$\frac{\Theta \longrightarrow \Theta'}{\Theta, \alpha \longrightarrow \Theta', \alpha}\ \textsf{Cuvar}$$

$\Theta \Longrightarrow \Theta'$ By i.h.

☞ $\Theta, \alpha \Longrightarrow \Theta', \alpha$ By Wcuvar

- **Case**
$$\frac{\Theta \longrightarrow \Theta'}{\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha}}\ \textsf{Cunsolvedguess}$$

$\Theta \Longrightarrow \Theta'$ By i.h.

☞ $\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha}$ By Wcunsolvedguess

- **Case**
$$\frac{\Theta \longrightarrow \Theta'}{\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} = P}\ \textsf{Csolveguess}$$

$\Theta \Longrightarrow \Theta'$ By i.h.

☞ $\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha} = P$ By Wcsolveguess

- **Case**
$$\frac{\Theta \longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \longrightarrow \Theta', \hat{\alpha} = Q}\ \textsf{Csolvedguess}$$

$\Theta \Longrightarrow \Theta'$ By i.h.

$\|\Theta\| \vdash P \cong^{+} Q$ Premise

☞ $\Theta, \hat{\alpha} = P \Longrightarrow \Theta', \hat{\alpha} = P$ By Wcsolvedguess

□

Lemma C.2 (Weak context extension is reflexive)**.** *For all contexts* Θ*,* $\Theta \Longrightarrow \Theta$*.*

Proof. Corollary of Lemma D.1 (Context extension is reflexive).

$\Theta \longrightarrow \Theta$ By Lemma D.1 (Context extension is reflexive)
☞ $\Theta \Longrightarrow \Theta$ By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)

□

Lemma C.3 (Equality of declarative contexts (weak)). *If* $\Theta \Longrightarrow \Theta'$*, then* $\|\Theta\| = \|\Theta'\|$.

Proof. By rule induction over the $\Theta \Longrightarrow \Theta'$ judgment.

- **Case**
$$\frac{}{\cdot \Longrightarrow \cdot}\ \textsf{Wcempty}$$

☞ $\|\cdot\| = \|\cdot\|$

- **Case**
$$\frac{\Theta \Longrightarrow \Theta'}{\Theta, \alpha \Longrightarrow \Theta', \alpha}\ \textsf{Wcuvar}$$

$$\begin{aligned} \|\Theta, \alpha\| &= \|\Theta\|, \alpha && \text{By definition of } \|-\| \\ &= \|\Theta'\|, \alpha && \text{By i.h.} \\ &= \|\Theta', \alpha\| && \text{By definition of } \|-\| \end{aligned}$$

- **Case**
$$\frac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha}}\ \textsf{Wcunsolvedguess}$$

$$\begin{aligned} \|\Theta, \hat{\alpha}\| &= \|\Theta\| && \text{By definition of } \|-\| \\ &= \|\Theta'\| && \text{By i.h.} \\ &= \|\Theta', \hat{\alpha}\| && \text{By definition of } \|-\| \end{aligned}$$

- **Case**
$$\frac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha} = P}\ \textsf{Wcsolveguess}$$

$$\begin{aligned} \|\Theta, \hat{\alpha}\| &= \|\Theta\| && \text{By definition of } \|-\| \\ &= \|\Theta'\| && \text{By i.h.} \\ &= \|\Theta', \hat{\alpha} = P\| && \text{By definition of } \|-\| \end{aligned}$$

- **Case**
$$\frac{\Theta \Longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \Longrightarrow \Theta', \hat{\alpha} = Q}\ \textsf{Wcsolvedguess}$$

$$\begin{aligned}\|\Theta, \hat{\alpha} = P\| &= \|\Theta\| && \text{By definition of } \|-\| \\ &= \|\Theta'\| && \text{By i.h.} \\ &= \|\Theta', \hat{\alpha} = Q\| && \text{By definition of } \|-\|\end{aligned}$$

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha}}$ Wcnewunsolvedguess

$$\begin{aligned}\|\Theta\| &= \|\Theta'\| && \text{By i.h.} \\ &= \|\Theta', \hat{\alpha}\| && \text{By definition of } \|-\|\end{aligned}$$

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha} = P}$ Wcnewsolvedguess

$$\begin{aligned}\|\Theta\| &= \|\Theta'\| && \text{By i.h.} \\ &= \|\Theta', \hat{\alpha} = P\| && \text{By definition of } \|-\|\end{aligned}$$

□

Lemma C.4 (Weak context extension is transitive). *If* $\Theta \Longrightarrow \Theta'$ *and* $\Theta' \Longrightarrow \Theta''$*, then* $\Theta \Longrightarrow \Theta''$*.*

Proof. By rule induction over the $\Theta' \Longrightarrow \Theta''$ judgment.

- Neither Wcnewunsolvedguess nor Wcnewunsolvedguess:
  By rule induction over the $\Theta \Longrightarrow \Theta'$ judgment.

  - **Case** $\dfrac{}{\cdot \Longrightarrow \cdot}$ Wcempty

☞ $\cdot \Longrightarrow \Theta''$ Assumption

  - **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \alpha \Longrightarrow \Theta', \alpha}$ Wcuvar

    By inversion on the second assumption (Wcuvar), the last context must be Θ'', α.

$$\begin{aligned}\Theta' &\Longrightarrow \Theta'' && \text{Inversion (Wcuvar)} \\ \Theta &\Longrightarrow \Theta'' && \text{By i.h.}\end{aligned}$$

☞ $\Theta, \alpha \Longrightarrow \Theta'', \alpha$ By Wcuvar

– **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha}}$ Wcunsolvedguess

By inversion on the second assumption, we must have either $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha}$ (Wcunsolvedguess) or $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha} = P$ (Wcsolveguess):

* **Case** $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha}$:

	$\Theta' \Longrightarrow \Theta''$	Inversion (Wcunsolvedguess)
	$\Theta \Longrightarrow \Theta''$	By i.h.
☞	$\Theta, \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha}$	By Wcunsolvedguess

* **Case** $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha} = P$:

	$\Theta' \Longrightarrow \Theta''$	Inversion (Wcsolveguess)
	$\Theta \Longrightarrow \Theta''$	By i.h.
☞	$\Theta, \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha} = P$	By Wcsolveguess

– **Case** $\dfrac{\Theta \Longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \Longrightarrow \Theta', \hat{\alpha} = Q}$ Wcsolvedguess

By inversion on the second assumption (Wcsolvedguess), the last context must be of the form $\Theta'', \hat{\alpha} = R$.

	$\Theta', \hat{\alpha} = Q \Longrightarrow \Theta'', \hat{\alpha} = R$	Assumption
	$\Theta' \Longrightarrow \Theta''$	Inversion (Wcsolvedguess)
	$\|\Theta'\| \vdash Q \cong^{+} R$	″
	$\Theta \Longrightarrow \Theta''$	By i.h.
	$\|\Theta\| \vdash P \cong^{+} Q$	Premise
	$\|\Theta\| \vdash Q \cong^{+} R$	By Lemma C.3 (Equality of declarative contexts (weak))
	$\|\Theta\| \vdash P \cong^{+} R$	By Lemma B.7 (Declarative subtyping is transitive)
☞	$\Theta, \hat{\alpha} = P \Longrightarrow \Theta'', \hat{\alpha} = R$	By Wcsolvedguess

– **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha} = P}$ Wcsolveguess

By inversion on the second assumption (Wcsolvedguess), the last context must be of the form $\Theta'', \hat{\alpha} = Q$.

	$\Theta', \hat{\alpha} = P \Longrightarrow \Theta'', \hat{\alpha} = Q$	Assumption
	$\Theta' \Longrightarrow \Theta''$	Inversion (Wcsolvedguess)
	$\Theta \Longrightarrow \Theta''$	By i.h.
☞	$\Theta, \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha} = Q$	By Wcsolveguess

– **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha}}$ Wcnewunsolvedguess

By inversion on the second assumption, we must have either $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha}$ (Wcunsolvedguess) or $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha} = P$ (Wcsolvedguess):

* **Case** $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha}$:

$\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha}$ Assumption
$\Theta' \Longrightarrow \Theta''$ Inversion (Wcunsolvedguess)

$\Theta \Longrightarrow \Theta''$ By i.h.
$\Theta \Longrightarrow \Theta'', \hat{\alpha}$ By Wcnewunsolvedguess

* **Case** $\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha} = P$:

$\Theta', \hat{\alpha} \Longrightarrow \Theta'', \hat{\alpha} = P$ Assumption
$\Theta' \Longrightarrow \Theta''$ Inversion (Wcsolvedguess)

$\Theta \Longrightarrow \Theta''$ By i.h.
$\Theta \Longrightarrow \Theta'', \hat{\alpha} = P$ By Wcnewsolvedguess

– **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha} = P}$ Wcnewsolvedguess

By inversion on the second assumption (Wcsolvedguess), the last context must be of the form $\Theta'', \hat{\alpha} = Q$.

$\Theta', \hat{\alpha} = P \Longrightarrow \Theta'', \hat{\alpha} = Q$ Assumption
$\Theta' \Longrightarrow \Theta''$ Inversion (Wcsolvedguess)

$\Theta \Longrightarrow \Theta''$ By i.h.
$\Theta \Longrightarrow \Theta'', \hat{\alpha} = Q$ By Wcnewsolvedguess

- **Case** $\dfrac{\Theta' \Longrightarrow \Theta''}{\Theta' \Longrightarrow \Theta'', \hat{\alpha}}$ Wcnewunsolvedguess

$\Theta \Longrightarrow \Theta''$ By i.h.
☞ $\Theta \Longrightarrow \Theta'', \hat{\alpha}$ By Wcnewunsolvedguess

- **Case** $\dfrac{\Theta' \Longrightarrow \Theta''}{\Theta' \Longrightarrow \Theta'', \hat{\alpha} = P}$ Wcnewunsolvedguess

	$\Theta \Longrightarrow \Theta''$	By i.h.
☞	$\Theta \Longrightarrow \Theta'', \hat{\alpha} = P$	By Wcnewsolvedguess

□

Lemma C.5 (Weak context extension preserves well-formedness). *If* $\Theta \vdash A\ \mathsf{type}^{\pm}$ *and* $\Theta \Longrightarrow \Theta'$ *then* $\Theta' \vdash A\ \mathsf{type}^{\pm}$.

Proof. By rule induction over the $\Theta \vdash A\ \mathsf{type}^{\pm}$ judgment.

- **Case** $\dfrac{\alpha \in FUV(\Theta)}{\Theta \vdash \alpha\ \mathsf{type}^{+}}$ Twfuvar

	$\alpha \in FUV(\Theta)$	Premise
	$\alpha \in FUV(\Theta')$	Inversion (must have instance of Wcuvar)
☞	$\Theta' \vdash \alpha\ \mathsf{type}^{+}$	By Twfuvar

- **Case** $\dfrac{\hat{\alpha} \in FEV(\Theta)}{\Theta \vdash \hat{\alpha}\ \mathsf{type}^{+}}$ Twfguess

	$\hat{\alpha} \in FEV(\Theta)$	Premise
	$\hat{\alpha} \in FEV(\Theta')$	Inversion (must have instance of Wcunsolvedguess, Wcsolveguess, or Wcsolvedguess)
☞	$\Theta' \vdash \hat{\alpha}\ \mathsf{type}^{+}$	By Twfguess

- **Case** $\dfrac{\Theta \vdash N\ \mathsf{type}^{-}}{\Theta \vdash {\downarrow}N\ \mathsf{type}^{+}}$ Twfshift↓

	$\Theta' \vdash N\ \mathsf{type}^{-}$	By i.h.
☞	$\Theta' \vdash {\downarrow}N\ \mathsf{type}^{-}$	By Twfshift↓

- **Case** $\dfrac{\Theta, \alpha \vdash N\ \mathsf{type}^{-}}{\Theta \vdash \forall\alpha.\, N\ \mathsf{type}^{-}}$ Twfforall

	$\Theta \Longrightarrow \Theta'$	Assumption
	$\Theta, \alpha \Longrightarrow \Theta', \alpha$	By Wcuvar
	$\Theta', \alpha \vdash N\ \mathsf{type}^{-}$	By i.h.
☞	$\Theta' \vdash \forall\alpha.\, N\ \mathsf{type}^{-}$	By Twfforall

- **Case** $\dfrac{\Theta \vdash P\ \mathsf{type}^{+} \qquad \Theta \vdash N\ \mathsf{type}^{-}}{\Theta \vdash P \to N\ \mathsf{type}^{-}}$ Twfarrow

	$\Theta' \vdash P\ \mathsf{type}^{+}$	By i.h.
	$\Theta' \vdash N\ \mathsf{type}^{-}$	By i.h.
☞	$\Theta' \vdash P \to N\ \mathsf{type}^{-}$	By Twfarrow

- **Case** $\dfrac{\Theta \vdash P\ \mathsf{type}^{+}}{\Theta \vdash {\uparrow}P\ \mathsf{type}^{-}}$ Twfshift↑

$\Theta' \vdash P\ \mathsf{type}^{+}$	By i.h.
$\Theta' \vdash {\uparrow}P\ \mathsf{type}^{-}$	By Twfshift↑

□

Lemma C.6 (Weak context extension preserves w.f. envs)**.** *If* $\Theta \implies \Theta'$ *and* $\Theta \vdash \Gamma\ \mathsf{env}$*, then* $\Theta' \vdash \Gamma\ \mathsf{env}$.

Proof. By rule induction over the definition of well-formed typing environments.

- **Case** $\dfrac{}{\Theta \vdash \cdot\ \mathsf{env}}$ Ewfempty

☞	$\Theta' \vdash \cdot\ \mathsf{env}$	By Ewfempty

- **Case** $\dfrac{\Theta \vdash \Gamma\ \mathsf{env} \qquad \Theta \vdash P\ \mathsf{type}^{+} \qquad P\ \mathsf{ground}}{\Theta \vdash \Gamma, x : P\ \mathsf{env}}$ Ewfvar

	$\Theta \vdash \Gamma, x : P\ \mathsf{env}$	Assumption
	$\Theta \vdash \Gamma\ \mathsf{env}$	By premise
	$\Theta' \vdash \Gamma\ \mathsf{env}$	By i.h.
	$\Theta \vdash P\ \mathsf{type}^{\pm}$	By premise
	$\Theta' \vdash P\ \mathsf{type}^{\pm}$	By Lemma C.5
	$P\ \mathsf{ground}$	By premise
☞	$\Theta' \vdash \Gamma, x : P\ \mathsf{env}$	By Ewfvar

□

Lemma C.7 (The extended context makes the type ground (weak)). *If* Θ ctx*,* Θ' ctx*,* $\Theta \Longrightarrow \Theta'$*, and* $[\Theta'][\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

Proof. Consider an arbitrary existential variable $\hat{\alpha}$ in A. Then for $[\Theta'][\Theta]A$ to be ground, we must have at least one of $\hat{\alpha} = P \in \Theta$, or $\hat{\alpha} = Q \in \Theta'$. We know that applying the contexts to the type will never introduce a non-ground type since Θ ctx and Θ' ctx.

By inversion on $\Theta \Longrightarrow \Theta'$, we can also see that if an existential variable is solved in the left-hand side context, then it must also be solved in the right-hand side context. Therefore we must have that $\hat{\alpha} = Q \in \Theta'$, and by Θ' ctx we know that Q is ground.

We now know that every existential variable in A is solved as a ground type by Θ', hence $[\Theta']A$ must be ground. □

Lemma C.8 (Extending context preserves groundness (weak)). *If* Θ ctx*,* Θ' ctx*,* $\Theta \Longrightarrow \Theta'$*, and* $[\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

Proof. Corollary of Lemma C.7 (The extended context makes the type ground (weak)).

	Θ ctx	Assumption
	Θ' ctx	Assumption
	$\Theta \Longrightarrow \Theta'$	Assumption
	$[\Theta]A$ ground	Assumption
	$[\Theta'][\Theta]A$ ground	By Lemma D.5 (Applying a context to a ground type)
☞	$[\Theta']A$ ground	By Lemma C.7 (The extended context makes the type ground (weak))

□

D' Context extension

Lemma D.1 (Context extension is reflexive). *For all contexts* Θ*,* $\Theta \longrightarrow \Theta$*.*

Proof. By structural induction on Θ.

- **Case** $\cdot$:

☞	$\cdot \longrightarrow \cdot$	By Cempty

- **Case** Θ, α:

	$\Theta \longrightarrow \Theta$	By i.h.
☞	$\Theta, \alpha \longrightarrow \Theta, \alpha$	By Cuvar

- **Case** $\Theta, \hat{\alpha}$:

	$\Theta \longrightarrow \Theta$	By i.h.
☞	$\Theta, \hat{\alpha} \longrightarrow \Theta, \hat{\alpha}$	By Cunsolvedguess

- **Case** $\Theta, \hat{\alpha} = P$:

$$\begin{array}{lll} & \Theta \longrightarrow \Theta & \text{By i.h.} \\ & \Theta \vdash P \cong^{+} P & \text{By Lemma B.1 (Declarative subtyping is reflexive)} \\ ☞ & \Theta, \hat{\alpha} = P \longrightarrow \Theta, \hat{\alpha} = P & \text{By Csolvedguess} \end{array}$$

□

Lemma D.2 (Equality of declarative contexts). *If* $\Theta \longrightarrow \Theta'$*, then* $\|\Theta\| = \|\Theta'\|$.

Proof. Corollary of Lemma C.3 (Equality of declarative contexts (weak)).

$$\begin{array}{lll} & \Theta \longrightarrow \Theta' & \text{Assumption} \\ & \Theta \Longrightarrow \Theta' & \text{By Lemma C.1 } (\Longrightarrow \text{ subsumes } \longrightarrow) \\ ☞ & \|\Theta\| = \|\Theta'\| & \text{By Lemma C.3 (Equality of declarative contexts (weak))} \end{array}$$

□

Lemma D.3 (Context extension is transitive). *If* $\Theta \longrightarrow \Theta'$ *and* $\Theta' \longrightarrow \Theta''$*, then* $\Theta \longrightarrow \Theta''$.

Proof. By rule induction over the $\Theta \longrightarrow \Theta'$ judgment.

- **Case** $\dfrac{}{\cdot \longrightarrow \cdot}$ Cempty

$$\begin{array}{ll} ☞ \quad \cdot \longrightarrow \Theta'' & \text{Assumption} \end{array}$$

- **Case** $\dfrac{\Theta \longrightarrow \Theta'}{\Theta, \alpha \longrightarrow \Theta', \alpha}$ Cuvar

  By inversion on the second assumption (Cuvar), the last context must be Θ'', α.

$$\begin{array}{lll} & \Theta' \longrightarrow \Theta'' & \text{Inversion (Cuvar)} \\ & \Theta \longrightarrow \Theta'' & \text{By i.h.} \\ ☞ & \Theta, \alpha \longrightarrow \Theta'', \alpha & \text{By Cuvar} \end{array}$$

- **Case** $\dfrac{\Theta \longrightarrow \Theta'}{\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha}}$ Cunsolvedguess

  By inversion on the second assumption, we must have either $\Theta', \hat{\alpha} \longrightarrow \Theta'', \hat{\alpha}$ (Cunsolvedguess) or $\Theta', \hat{\alpha} \longrightarrow \Theta'', \hat{\alpha} = P$ (Csolveguess):

  - **Case** $\Theta', \hat{\alpha} \longrightarrow \Theta'', \hat{\alpha}$:

	$\Theta' \longrightarrow \Theta''$	Inversion (Cunsolvedguess)
	$\Theta \longrightarrow \Theta''$	By i.h.
☞	$\Theta, \hat{\alpha} \longrightarrow \Theta'', \hat{\alpha}$	By Cunsolvedguess

– **Case** $\Theta', \hat{\alpha} \longrightarrow \Theta'', \hat{\alpha} = P$:

	$\Theta' \longrightarrow \Theta''$	Inversion (Csolveguess)
	$\Theta \longrightarrow \Theta''$	By i.h.
☞	$\Theta, \hat{\alpha} \longrightarrow \Theta'', \hat{\alpha} = P$	By Csolveguess

- **Case** $\dfrac{\Theta \longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \longrightarrow \Theta', \hat{\alpha} = Q}$ Csolvedguess

  By inversion on the second assumption (Csolvedguess), the last context must be of the form $\Theta'', \hat{\alpha} = R$.

	$\Theta', \hat{\alpha} = Q \longrightarrow \Theta'', \hat{\alpha} = R$	Assumption
	$\Theta' \longrightarrow \Theta''$	Inversion (Csolvedguess)
	$\|\Theta'\| \vdash Q \cong^{+} R$	″
	$\Theta \longrightarrow \Theta''$	By i.h.
	$\|\Theta\| \vdash P \cong^{+} Q$	Premise
	$\|\Theta\| \vdash Q \cong^{+} R$	By Lemma D.2 (Equality of declarative contexts)
	$\|\Theta\| \vdash P \cong^{+} R$	By Lemma B.7 (Declarative subtyping is transitive)
☞	$\Theta, \hat{\alpha} = P \longrightarrow \Theta'', \hat{\alpha} = R$	By Csolvedguess

- **Case** $\dfrac{\Theta \longrightarrow \Theta'}{\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} = P}$ Csolveguess

  By inversion on the second assumption (Csolvedguess), the last context must be of the form $\Theta'', \hat{\alpha} = Q$.

	$\Theta', \hat{\alpha} = P \longrightarrow \Theta'', \hat{\alpha} = Q$	Assumption
	$\Theta' \longrightarrow \Theta''$	Inversion (Csolvedguess)
	$\Theta \longrightarrow \Theta''$	By i.h.
☞	$\Theta, \hat{\alpha} \longrightarrow \Theta'', \hat{\alpha} = Q$	By Csolveguess

□

Lemma D.4 (Context extension preserves w.f.). *If* $\Theta \vdash A \text{ type}^{\pm}$ *and* $\Theta \longrightarrow \Theta'$*, then* $\Theta' \vdash A \text{ type}^{\pm}$.

Proof. By rule induction on $\Theta \vdash A \text{ type}^{\pm}$.

- **Case** $\dfrac{\alpha \in \mathsf{UV}(\Theta)}{\Theta \vdash \alpha\ \mathsf{type}^+}$ Twfuvar

$$\begin{array}{rll}
 & \alpha \in \mathsf{UV}(\Theta) & \text{Subderivation} \\
 & \Theta \longrightarrow \Theta' & \text{Assumption} \\
 & \alpha \in \mathsf{UV}(\Theta') & \text{Inversion (Cuvar)} \\
☞ & \Theta' \vdash \alpha\ \mathsf{type}^+ & \text{By Twfuvar}
\end{array}$$

- **Case** $\dfrac{\hat{\alpha} \in \mathsf{EV}(\Theta)}{\Theta \vdash \hat{\alpha}\ \mathsf{type}^+}$ Twfguess

$$\begin{array}{rll}
 & \Theta \longrightarrow \Theta' & \text{Assumption} \\
 & \hat{\alpha} \in \mathsf{EV}(\Theta) & \text{Subderivation} \\
 & \hat{\alpha} \in \mathsf{EV}(\Theta') & \text{Must have an instance of Cunsolvedguess, Csolveguess, or Csolvedguess} \\
☞ & \Theta' \vdash \hat{\alpha}\ \mathsf{type}^+ & \text{By Twfuvar}
\end{array}$$

- **Case** $\dfrac{\Theta \vdash \mathsf{N}\ \mathsf{type}^-}{\Theta \vdash {\downarrow}\mathsf{N}\ \mathsf{type}^+}$ Twfshift↓

$$\begin{array}{rll}
 & \Theta \vdash \mathsf{N}\ \mathsf{type}^- & \text{Subderivation} \\
 & \Theta \longrightarrow \Theta' & \text{Assumption} \\
 & \Theta' \vdash \mathsf{N}\ \mathsf{type}^- & \text{By i.h.} \\
☞ & \Theta' \vdash {\downarrow}\mathsf{N}\ \mathsf{type}^+ & \text{By Twfshift}{\downarrow}
\end{array}$$

- **Case** $\dfrac{\Theta, \alpha \vdash \mathsf{N}\ \mathsf{type}^-}{\Theta \vdash \forall\alpha.\, \mathsf{N}\ \mathsf{type}^-}$ Twfforall

$$\begin{array}{rll}
 & \Theta, \alpha \vdash \mathsf{N}\ \mathsf{type}^- & \text{Subderivation} \\
 & \Theta \longrightarrow \Theta' & \text{Assumption} \\
 & \Theta, \alpha \longrightarrow \Theta', \alpha & \text{By Cuvar} \\
 & \Theta', \alpha \vdash \mathsf{N}\ \mathsf{type}^- & \text{By i.h.} \\
☞ & \Theta' \vdash \forall\alpha.\, \mathsf{N}\ \mathsf{type}^- & \text{By Twfforall}
\end{array}$$

- **Case** $\dfrac{\Theta \vdash \mathsf{P}\ \mathsf{type}^+ \qquad \Theta \vdash \mathsf{N}\ \mathsf{type}^-}{\Theta \vdash \mathsf{P} \to \mathsf{N}\ \mathsf{type}^-}$ Twfarrow

$$\begin{array}{ll}
\Theta \vdash \mathsf{P}\ \mathsf{type}^+ & \text{Subderivation} \\
\Theta \longrightarrow \Theta' & \text{Assumption} \\
\Theta' \vdash \mathsf{P}\ \mathsf{type}^+ & \text{By i.h.}
\end{array}$$

	$\Theta \vdash N \;\; \mathsf{type}^-$	Subderivation
	$\Theta' \vdash N \;\; \mathsf{type}^-$	By i.h.
☞	$\Theta' \vdash P \to N \;\; \mathsf{type}^-$	By Twfarrow

- **Case** $\dfrac{\Theta \vdash P\, \mathsf{type}^+}{\Theta \vdash {\uparrow}P\, \mathsf{type}^-}$ Twfshift↑

	$\Theta \vdash P \;\; \mathsf{type}^+$	Subderivation
	$\Theta \longrightarrow \Theta'$	"
	$\Theta' \vdash P \;\; \mathsf{type}^+$	By i.h.
☞	$\Theta' \vdash {\uparrow}P \;\; \mathsf{type}^-$	By Twfshift↑

□

Lemma D.5 (Applying a context to a ground type)**.** *If* A *ground, then* $[\Theta]A = A$.

Proof. By structural induction on Θ.

- **Case** $\cdot$:

$[\cdot]A = A$ By definition of $[-]-$

- **Case** Θ, α:

$$
\begin{aligned}
[\Theta, \alpha]A &= [\Theta]A && \text{By definition of } [-]- \\
&= A && \text{By i.h.}
\end{aligned}
$$

- **Case** $\Theta, \hat{\alpha}$:

$$
\begin{aligned}
[\Theta, \hat{\alpha}]A &= [\Theta]A && \text{By definition of } [-]- \\
&= A && \text{By i.h.}
\end{aligned}
$$

- **Case** $\Theta, \hat{\alpha} = P$:

$$
\begin{aligned}
[\Theta, \hat{\alpha} = P]A &= [\Theta]([P/\hat{\alpha}]A) && \text{By definition of } [-]- \\
&= [\Theta]A && A \text{ ground, so no } \hat{\alpha}\text{s to substitute} \\
&= A && \text{By i.h.}
\end{aligned}
$$

□

Lemma D.6 (Context application is idempotent)**.** *If* Θ ctx*, then* $[\Theta][\Theta]A = [\Theta]A$.

Proof. By structural induction on Θ:.

- **Case** $\cdot$:

$$[\cdot][\cdot]A = A \quad \text{By definition of } [-]-$$

- **Case** Θ, α:

$$\begin{aligned} [\Theta, \alpha][\Theta, \alpha]A &= [\Theta][\Theta]A && \text{By definition of } [-]- \\ &= A && \text{By i.h.} \end{aligned}$$

- **Case** $\Theta, \hat{\alpha}$:

$$\begin{aligned} [\Theta, \hat{\alpha}][\Theta, \hat{\alpha}]A &= [\Theta][\Theta]A && \text{By definition of } [-]- \\ &= A && \text{By i.h.} \end{aligned}$$

- **Case** $\Theta, \hat{\alpha} = P$:

$$\begin{aligned} [\Theta, \hat{\alpha} = P][\Theta, \hat{\alpha} = P]A &= [\Theta][P/\hat{\alpha}][\Theta][P/\hat{\alpha}]A && \text{By definition of } [-]- \\ &= [P/\hat{\alpha}][\Theta][\Theta]A && P \text{ ground and } \Theta, \hat{\alpha} = P \text{ ctx, so } \hat{\alpha} \text{ does not reappear} \\ &= [P/\hat{\alpha}][\Theta]A && \text{By i.h.} \\ &= [\Theta][P/\hat{\alpha}]A && P \text{ ground and } \Theta, \hat{\alpha} = P \text{ ctx, so } \hat{\alpha} \text{ does not reappear} \\ &= [\Theta, \hat{\alpha} = P]A && \text{By definition of } [-]- \end{aligned}$$

□

Lemma D.7 (The extended context makes the type ground)**.** *If* Θ ctx*,* Θ' ctx*,* $\Theta \longrightarrow \Theta'$*, and* $[\Theta'][\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

Proof. Corollary of Lemma C.7 (The extended context makes the type ground (weak)).

$$\begin{aligned} \Theta \ &\text{ctx} && \text{Assumption} \\ \Theta' \ &\text{ctx} && \text{Assumption} \\ \Theta &\longrightarrow \Theta' && \text{Assumption} \\ \Theta &\Longrightarrow \Theta' && \text{By Lemma C.1 } (\Longrightarrow \text{ subsumes } \longrightarrow) \\ [\Theta'][\Theta]A \ &\text{ground} && \text{Assumption} \\ [\Theta']A \ &\text{ground} && \text{By Lemma C.7 (The extended context makes the type ground (weak))} \end{aligned}$$

□

Lemma D.8 (Extending context preserves groundness)**.** *If* Θ ctx*,* Θ' ctx*,* $\Theta \longrightarrow \Theta'$*, and* $[\Theta]A$ *ground, then* $[\Theta']A$ *ground.*

Proof. Corollary of Lemma C.8 (Extending context preserves groundness (weak)).

Θ ctx	Assumption
Θ' ctx	Assumption
$\Theta \longrightarrow \Theta'$	Assumption
$\Theta \Longrightarrow \Theta'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
$[\Theta]A$ ground	Assumption
$[\Theta']A$ ground	By Lemma C.8 (Extending context preserves groundness (weak))

□

E' Well-formedness of subtyping

Lemma E.1 (Applying context to the type preserves w.f.). *If* Θ ctx *and* $\Theta \vdash A\ \text{type}^{\pm}$*, then* $\Theta \vdash [\Theta]A\ \text{type}^{\pm}$.

Proof. By rule induction on $\Theta \vdash A\ \text{type}^{\pm}$.

- **Case** $\dfrac{\alpha \in \text{UV}(\Theta)}{\Theta \vdash \alpha\ \text{type}^{+}}\ \text{Twfuvar}$

	$\Theta \vdash \alpha\ \text{type}^{+}$	Assumption
	$[\Theta]\alpha = \alpha$	By definition of $[-]-$
☞	$\Theta \vdash [\Theta]\alpha\ \text{type}^{+}$	By above two statements

- **Case** $\dfrac{\hat{\alpha} \in \text{EV}(\Theta)}{\Theta \vdash \hat{\alpha}\ \text{type}^{+}}\ \text{Twfguess}$

Case $(\hat{\alpha} = P) \in \Theta$:

	Θ ctx	Assumption
	$\Theta \vdash P\ \text{type}^{+}$	Must have an instance of Cwfsolvedguess
	$[\Theta]\hat{\alpha} = P$	By definition of $[-]-$
☞	$\Theta \vdash [\Theta]\hat{\alpha}\ \text{type}^{+}$	By above two statements

Case $(\hat{\alpha} = P) \notin \Theta$:

	$\Theta \vdash \hat{\alpha}\ \text{type}^{+}$	Assumption
	$[\Theta]\hat{\alpha} = \hat{\alpha}$	Since $(\hat{\alpha} = P) \notin \Theta$
☞	$\Theta \vdash [\Theta]\hat{\alpha}\ \text{type}^{+}$	By above two statements

- **Case** $\dfrac{\Theta \vdash N\ \text{type}^{-}}{\Theta \vdash \downarrow N\ \text{type}^{+}}\ \text{Twfshift}\downarrow$

Θ ctx	Assumption

$$\begin{array}{rll}
& \Theta \vdash N \;\; \mathsf{type}^- & \text{Subderivation} \\
& \Theta \vdash [\Theta]N \;\; \mathsf{type}^- & \text{By i.h.} \\
& \Theta \vdash {\downarrow}[\Theta]N \;\; \mathsf{type}^+ & \text{By Twfshift}{\downarrow} \\
☞ & \Theta \vdash [\Theta]{\downarrow}N \;\; \mathsf{type}^+ & \text{By definition of } [-]-
\end{array}$$

- **Case** $\dfrac{\Theta, \alpha \vdash N \;\mathsf{type}^-}{\Theta \vdash \forall\alpha.\, N \;\mathsf{type}^-}\;\mathsf{Twfforall}$

$$\begin{array}{rll}
& \Theta \;\; \mathsf{ctx} & \text{Assumption} \\
& \Theta, \alpha \;\; \mathsf{ctx} & \text{By Cwfuvar} \\
& \Theta, \alpha \vdash N \;\; \mathsf{type}^- & \text{Subderivation} \\
& \Theta, \alpha \vdash [\Theta, \alpha]N \;\; \mathsf{type}^- & \text{By i.h.} \\
& \Theta, \alpha \vdash [\Theta]N \;\; \mathsf{type}^- & \text{By definition of } [-]- \\
& \Theta \vdash \forall\alpha.\, [\Theta]N \;\; \mathsf{type}^- & \text{By Twfforall} \\
☞ & \Theta \vdash [\Theta]\forall\alpha.\, N \;\; \mathsf{type}^- & \text{By definition of } [-]-
\end{array}$$

- **Case** $\dfrac{\Theta \vdash P \;\mathsf{type}^+ \qquad \Theta \vdash N \;\mathsf{type}^-}{\Theta \vdash P \to N \;\mathsf{type}^-}\;\mathsf{Twfarrow}$

$$\begin{array}{rll}
& \Theta \;\; \mathsf{ctx} & \text{Assumption} \\
& \Theta \vdash P \;\; \mathsf{type}^+ & \text{Subderivation} \\
& \Theta \vdash [\Theta]P \;\; \mathsf{type}^+ & \text{By i.h.} \\
& \Theta \vdash N \;\; \mathsf{type}^- & \text{Subderivation} \\
& \Theta \vdash [\Theta]N \;\; \mathsf{type}^- & \text{By i.h.} \\
& \Theta \vdash [\Theta]P \to [\Theta]N \;\; \mathsf{type}^- & \text{By Twfarrow} \\
☞ & \Theta \vdash [\Theta](P \to N) \;\; \mathsf{type}^- & \text{By definition of } [-]-
\end{array}$$

- **Case** $\dfrac{\Theta \vdash P \;\mathsf{type}^+}{\Theta \vdash {\uparrow}P \;\mathsf{type}^-}\;\mathsf{Twfshift}{\uparrow}$

$$\begin{array}{rll}
& \Theta \;\; \mathsf{ctx} & \text{Assumption} \\
& \Theta \vdash P \;\; \mathsf{type}^+ & \text{Subderivation} \\
& \Theta \vdash [\Theta]P \;\; \mathsf{type}^+ & \text{By i.h.} \\
& \Theta \vdash {\uparrow}[\Theta]P \;\; \mathsf{type}^- & \text{By Twfshift}{\uparrow} \\
☞ & \Theta \vdash [\Theta]{\uparrow}P \;\; \mathsf{type}^- & \text{By definition of } [-]-
\end{array}$$

□

Lemma E.2 (Algorithmic subtyping is w.f.).

- *If* $\Theta \vdash P \leq^+ Q \dashv \Theta'$, Θ ctx*,* P *ground, and* $[\Theta]Q = Q$*, then* Θ' ctx*,* $\Theta \longrightarrow \Theta'$*, and* $[\Theta']Q$ *ground.*

- *If* $\Theta \vdash N \leq^- M \dashv \Theta'$, Θ ctx, M *ground, and* $[\Theta]N = N$, *then* Θ' ctx, $\Theta \longrightarrow \Theta'$, *and* $[\Theta']N$ *ground.*

Proof. By mutual induction on the derivation of $\Theta \vdash A \leq^\pm B \dashv \Theta'$.

- **Case**
$$\frac{}{\Theta_L, \alpha, \Theta_R \vdash \alpha \leq^+ \alpha \dashv \Theta_L, \alpha, \Theta_R} \leq^\pm\text{Arefl}$$

☞	$\Theta_L, \alpha, \Theta_R$ ctx	Assumption
☞	$\Theta_L, \alpha, \Theta_R \longrightarrow \Theta_L, \alpha, \Theta_R$	By Lemma D.1 (Context extension is reflexive)
	$[\Theta_L, \alpha, \Theta_R]\alpha = \alpha$	Assumption
	α ground	Assumption
☞	$[\Theta_L, \alpha, \Theta_R]\alpha$ ground	By the previous two statements

- **Case**
$$\frac{\Theta_L \vdash P \text{ type}^+ \qquad P \text{ ground}}{\Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^+ \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R} \leq^\pm\text{Ainst}$$

	$\Theta_L, \hat{\alpha}, \Theta_R$ ctx	Assumption
	$\Theta_L \vdash P$ type$^+$	Subderivation
	P ground	Assumption
☞	$\Theta_L, \hat{\alpha} = P, \Theta_R$ ctx	Replacing the instance of Cwfunsolvedguess corresponding to $\hat{\alpha}$ with an instance of Cwfsolvedguess
	$\Theta_L \longrightarrow \Theta_L$	By Lemma D.1 (Context extension is reflexive)
	$\Theta_L, \hat{\alpha} \longrightarrow \Theta_L, \hat{\alpha} = P$	By Csolveguess
	$\Theta_R \longrightarrow \Theta_R$	By Lemma D.1 (Context extension is reflexive)
	$\Theta_L, \hat{\alpha}, \Theta_R \longrightarrow \Theta_L, \hat{\alpha} = P, \Theta_R$	Reapplying rules from $\Theta_R \longrightarrow \Theta_R$
	$[\Theta_L, \hat{\alpha} = P, \Theta_R]\hat{\alpha} = P$	By definition of $[-]-$
☞	$[\Theta_L, \hat{\alpha} = P, \Theta_R]\hat{\alpha}$ ground	By the previous two statements

- **Case**
$$\frac{\Theta \vdash M \leq^- N \dashv \Theta' \qquad \Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''}{\Theta \vdash {\downarrow}N \leq^+ {\downarrow}M \dashv \Theta''} \leq^\pm\text{Ashift}{\downarrow}$$

${\downarrow}N$ ground	Assumption
$[\Theta]{\downarrow}M = {\downarrow}M$	Assumption

We have:

$\Theta \vdash M \leq^- N \dashv \Theta'$	Subderivation
Θ ctx	Assumption
N ground	By definition of ground
$[\Theta]M = M$	By definition of $[-]-$

Therefore:

	Θ' ctx	By i.h.
	$\Theta \longrightarrow \Theta'$	"
	$[\Theta']M$ ground	"

Now, looking at the second premise, we have:

$\Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''$	Subderivation
Θ' ctx	Above
$[\Theta']M$ ground	Above
$[\Theta']N = N$	By Lemma D.5 (Applying a context to a ground type)

Therefore:

	$\Theta' \longrightarrow \Theta''$	By i.h.
☞	Θ'' ctx	"
☞	$\Theta \longrightarrow \Theta''$	By Lemma D.3 (Context extension is transitive)
	$[\Theta'']M$ ground	By Lemma D.8 (Extending context preserves groundness)
☞	$[\Theta'']{\downarrow}M$ ground	By definition of ground

- **Case** $$\frac{\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha}{\Theta \vdash N \leq^- \forall\alpha.\, M \dashv \Theta'}\ \leq^{\pm}\mathsf{Aforallr}$$

Θ ctx	Assumption
$\forall\alpha.\, M$ ground	Assumption
$[\Theta]N = N$	Assumption

We have:

$\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha$	Subderivation
Θ, α ctx	By Cwfuvar
M ground	By definition of ground
$[\Theta, \alpha]N = N$	Since $[\Theta, \alpha]N = [\Theta]N$ by definition of $[-]-$

Therefore:

	Θ', α ctx	By i.h.
	$\Theta, \alpha \longrightarrow \Theta', \alpha$	"
	$[\Theta', \alpha]N$ ground	"
☞	Θ' ctx	Inversion (Cwfuvar)
☞	$\Theta \longrightarrow \Theta'$	Inversion (Cuvar)
☞	$[\Theta']N$ ground	Since $[\Theta']N = [\Theta', \alpha]N$ by definition of $[-]-$

- **Case** $$\frac{\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^- M \dashv \Theta', \hat{\alpha}\,[= P] \qquad M \neq \forall\alpha.\, M'}{\Theta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta'}\ \leq^{\pm}\mathsf{Aforalll}$$

$$\begin{array}{ll}
\Theta \ \text{ctx} & \text{Assumption} \\
[\Theta]\forall\alpha.\, N = \forall\alpha.\, N & \text{Assumption} \\
[\Theta]N = N & \text{By definition of } [-]-
\end{array}$$

We have:

$$\begin{array}{ll}
\Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N \leq^- M \dashv \Theta', \hat\alpha\,[=P] & \text{Subderivation} \\
\Theta, \hat\alpha \ \text{ctx} & \text{By Cwfunsolvedguess} \\
M \ \text{ground} & \text{Assumption} \\
[\Theta]\hat\alpha = \hat\alpha & \text{Since } \Theta, \hat\alpha \ \text{ctx} \\
[\Theta, \hat\alpha][\hat\alpha/\alpha]N = [\hat\alpha/\alpha]N & \text{Since } [\Theta]\hat\alpha = \hat\alpha \text{ and } [\Theta]N = N
\end{array}$$

Therefore:

$$\begin{array}{ll}
\Theta', \hat\alpha\,[=P] \ \text{ctx} & \text{By i.h.} \\
\Theta, \hat\alpha \longrightarrow \Theta', \hat\alpha\,[=P] & '' \\
[\Theta', \hat\alpha\,[=P]][\hat\alpha/\alpha]N \ \text{ground} & ''
\end{array}$$

$$\begin{array}{lll}
☞ & \Theta' \ \text{ctx} & \text{Inversion (Cwfuvar)} \\
☞ & \Theta \longrightarrow \Theta' & \text{Inversion (Cuvar)} \\
 & [\Theta']N \ \text{ground} & \text{Using above, } \alpha \text{ ground, and } \hat\alpha \notin \mathrm{FEV}(N) \\
☞ & [\Theta']\forall\alpha.\, N \ \text{ground} & \text{By definition of ground and } [-]-
\end{array}$$

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \dashv \Theta' \qquad \Theta' \vdash [\Theta']N \leq^- M \dashv \Theta''}{\Theta \vdash P \to N \leq^- Q \to M \dashv \Theta''} \leq^{\pm}\mathsf{Aarrow}$

$$\begin{array}{ll}
Q \to M \ \text{ground} & \text{Assumption} \\
[\Theta](P \to N) = P \to N & \text{Assumption}
\end{array}$$

We have:

$$\begin{array}{ll}
\Theta \vdash Q \leq^+ P \dashv \Theta' & \text{Subderivation} \\
\Theta \ \text{ctx} & \text{Assumption} \\
Q \ \text{ground} & \text{Since } Q \to M \text{ ground} \\
[\Theta]P = P & \text{By definition of } [-]-
\end{array}$$

Therefore:

$$\begin{array}{ll}
\Theta' \ \text{ctx} & \text{By i.h.} \\
\Theta \longrightarrow \Theta' & '' \\
[\Theta']P \ \text{ground} & ''
\end{array}$$

Looking at the second premise, we have:

$$\begin{array}{ll}
\Theta' \vdash [\Theta']N \leq^- M \dashv \Theta'' & \text{Subderivation} \\
\Theta' \ \text{ctx} & \text{Above} \\
M \ \text{ground} & \text{Since } Q \to M \text{ ground} \\
[\Theta'][\Theta']N = [\Theta']N & \text{By Lemma D.6 (Context application is idempotent)}
\end{array}$$

Therefore:

☞	Θ'' ctx	By i.h.
	$\Theta' \longrightarrow \Theta''$	"
	$[\Theta''][\Theta']N$ ground	"
☞	$\Theta \longrightarrow \Theta''$	By Lemma D.3 (Context extension is transitive)
	$[\Theta'']N$ ground	By Lemma D.7 (The extended context makes the type ground)
	$[\Theta'']P$ ground	Applying Lemma D.8 (Extending context preserves groundness) with $[\Theta']P$ ground
☞	$[\Theta'']P \to N$ ground	From equations above

- **Case** $$\frac{\Theta \vdash Q \leq^{+} P \dashv \Theta' \qquad \Theta' \vdash [\Theta']P \leq^{+} Q \dashv \Theta''}{\Theta \vdash \uparrow P \leq^{-} \uparrow Q \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}\uparrow$$

$\uparrow Q$ ground	Assumption
$[\Theta]\uparrow P = \uparrow P$	Assumption

We have:

$\Theta \vdash Q \leq^{+} P \dashv \Theta'$	Subderivation
Θ ctx	Assumption
Q ground	Since $\uparrow Q$ ground
$[\Theta]P = P$	By definition of $[-]-$

Therefore:

Θ' ctx	By i.h.
$\Theta \longrightarrow \Theta'$	"
$[\Theta']P$ ground	"

Looking at the second premise, we have:

$\Theta' \vdash [\Theta']P \leq^{+} Q \dashv \Theta''$	Subderivation
Θ' ctx	Above
$[\Theta']P$ ground	Above
$[\Theta']Q = Q$	By Lemma D.5 (Applying a context to a ground type)

Therefore:

☞	Θ'' ctx	By i.h.
	$\Theta' \longrightarrow \Theta''$	"
☞	$\Theta \longrightarrow \Theta''$	By Lemma D.3 (Context extension is transitive)
	$[\Theta'']P$ ground	Applying Lemma D.8 (Extending context preserves groundness) with $[\Theta']P$ ground
☞	$[\Theta'']\uparrow P$ ground	By definition of groundness and above

□

F' Soundness of subtyping

F'.1 Lemmas for soundness

Lemma F.1 (Completing context preserves w.f.). *If* $\Theta \vdash A\ \mathsf{type}^{\pm}$ *and* A *ground then* $\|\Theta\| \vdash A\ \mathsf{type}^{\pm}$.

Proof. By rule induction on $\Theta \vdash A\ \mathsf{type}^{\pm}$.

- **Case** $\dfrac{\alpha \in \mathrm{UV}(\Theta)}{\Theta \vdash \alpha\ \mathsf{type}^{+}}\ \mathsf{Twfuvar}$

	$\alpha \in \mathrm{UV}(\Theta)$	Subderivation
	$\alpha \in \mathrm{UV}([\Omega]\Theta)$	By definition of $[-]-$
☞	$\|\Theta\| \vdash \alpha\ \mathsf{type}^{+}$	By Twfuvar

- **Case** $\dfrac{\hat{\alpha} \in \mathrm{EV}(\Theta)}{\Theta \vdash \hat{\alpha}\ \mathsf{type}^{+}}\ \mathsf{Twfguess}$

  Not possible, since A is ground.

- **Case** $\dfrac{\Theta \vdash \mathsf{N}\ \mathsf{type}^{-}}{\Theta \vdash {\downarrow}\mathsf{N}\ \mathsf{type}^{+}}\ \mathsf{Twfshift}{\downarrow}$

	${\downarrow}\mathsf{N}$ ground	Assumption
	$\Theta \vdash \mathsf{N}\ \mathsf{type}^{-}$	Subderivation
	N ground	By definition of ground
	$\|\Theta\| \vdash \mathsf{N}\ \mathsf{type}^{-}$	By i.h.
☞	$\|\Theta\| \vdash {\downarrow}\mathsf{N}\ \mathsf{type}^{+}$	By Twfshift↓

- **Case** $\dfrac{\Theta, \alpha \vdash \mathsf{N}\ \mathsf{type}^{-}}{\Theta \vdash \forall \alpha.\, \mathsf{N}\ \mathsf{type}^{-}}\ \mathsf{Twfforall}$

	$\forall \alpha.\, \mathsf{N}$ ground	Assumption
	$\Theta, \alpha \vdash \mathsf{N}\ \mathsf{type}^{-}$	Subderivation
	N ground	By definition of ground
	$[\Omega](\Theta, \alpha) \vdash \mathsf{N}\ \mathsf{type}^{-}$	By i.h.
	$[\Omega]\Theta, \alpha \vdash \mathsf{N}\ \mathsf{type}^{-}$	By definition of $[-]-$
☞	$\|\Theta\| \vdash \forall \alpha.\, \mathsf{N}\ \mathsf{type}^{-}$	By Twfforall

- **Case** $\dfrac{\Theta \vdash \mathsf{P}\ \mathsf{type}^{+} \qquad \Theta \vdash \mathsf{N}\ \mathsf{type}^{-}}{\Theta \vdash \mathsf{P} \to \mathsf{N}\ \mathsf{type}^{-}}\ \mathsf{Twfarrow}$

$$
\begin{array}{rll}
& P \to N \ \text{ground} & \text{Assumption} \\
& \Theta \vdash P \ \text{type}^{+} & \text{Subderivation} \\
& P \ \text{ground} & \text{By definition of ground} \\
& \|\Theta\| \vdash P \ \text{type}^{+} & \text{By i.h.} \\
& \Theta \vdash N \ \text{type}^{-} & \text{Subderivation} \\
& N \ \text{ground} & \text{By definition of ground} \\
& \|\Theta\| \vdash N \ \text{type}^{-} & \text{By i.h.} \\
☞ & \|\Theta\| \vdash P \to N \ \text{type}^{-} & \text{By Twfarrow}
\end{array}
$$

- **Case** $\dfrac{\Theta \vdash P \ \text{type}^{+}}{\Theta \vdash \uparrow P \ \text{type}^{-}}$ Twfshift↑

$$
\begin{array}{rll}
& \uparrow P \ \text{ground} & \text{Assumption} \\
& \Theta \vdash P \ \text{type}^{+} & \text{Subderivation} \\
& P \ \text{ground} & \text{By definition of ground} \\
& \|\Theta\| \vdash P \ \text{type}^{+} & \text{By i.h.} \\
☞ & \|\Theta\| \vdash \uparrow P \ \text{type}^{-} & \text{By Twfshift}\uparrow
\end{array}
$$

□

Lemma F.2 ($\Longrightarrow$ leads to isomorphic types)**.** *If:*

1. $\Theta \vdash A \ \text{type}^{\pm}$
2. $\Theta \Longrightarrow \Theta'$
3. $[\Theta']A$ *ground*
4. $\Theta \ \text{ctx}$
5. $\Theta' \ \text{ctx}$

then $\|\Theta\| \vdash [\Theta'][\Theta]A \cong^{\pm} [\Theta']A$.

Proof. By rule induction on $\Theta \vdash A \ \text{type}^{\pm}$.

- **Case** $\dfrac{\alpha \in UV(\Theta)}{\Theta \vdash \alpha \ \text{type}^{+}}$ Twfuvar

$$
\begin{array}{rll}
& \Theta \vdash \alpha \ \text{type}^{+} & \text{Subderivation} \\
& \|\Theta\| \vdash \alpha \ \text{type}^{+} & \text{By Lemma F.1 (Completing context preserves w.f.)} \\
& \|\Theta\| \vdash \alpha \leq^{+} \alpha & \text{By } \leq^{\pm}\text{Drefl} \\
☞ & \|\Theta\| \vdash [\Theta'][\Theta]\alpha \cong^{+} [\Theta']\alpha & \text{By Lemma D.5 (Applying a context to a ground type)}
\end{array}
$$

- **Case** $\dfrac{\hat{\alpha} \in EV(\Theta)}{\Theta \vdash \hat{\alpha}\,\text{type}^+}$ Twfguess

Case $[\Theta]\hat{\alpha} = \hat{\alpha}$:

$(\hat{\alpha} = P) \in \Theta'$	Where $P = [\Theta']\hat{\alpha}$, since $[\Theta']\hat{\alpha}$ ground
$\Theta' \vdash P\ \text{type}^+$	Inversion on Θ' ctx (must have instance of Cwfsolvedguess)
$\|\Theta'\| \vdash P\ \text{type}^+$	By Lemma F.1 (Completing context preserves w.f.)
$\|\Theta\| \vdash P\ \text{type}^+$	By Lemma C.3 (Equality of declarative contexts (weak))
$\|\Theta\| \vdash [\Theta']\hat{\alpha}\ \text{type}^+$	Substituting for P
$\|\Theta\| \vdash [\Theta']\hat{\alpha} \cong^+ [\Theta']\hat{\alpha}$	By Lemma B.1 (Declarative subtyping is reflexive)
☞ $\|\Theta\| \vdash [\Theta'][\Theta]\hat{\alpha} \cong^+ [\Theta']\hat{\alpha}$	As we are in the case that $[\Theta]\hat{\alpha} = \hat{\alpha}$

Case $[\Theta]\hat{\alpha} \neq \hat{\alpha}$:

$\|\Theta_L\| \vdash [\Theta_L]\hat{\alpha} \cong^+ [\Theta'_L]\hat{\alpha}$	Inversion on $\Theta \Longrightarrow \Theta'$ (must have instance of Wcsolvedguess, Wcnewunsolvedguess, or Wcnewsolvedguess)
$\Theta_L \vdash [\Theta_L]\hat{\alpha}\ \text{type}^+$	Inversion on Θ ctx (must have instance of Cwfsolvedguess)
$\|\Theta\| \vdash [\Theta]\hat{\alpha} \cong^+ [\Theta']\hat{\alpha}$	By Lemma A.4 (Declarative subtyping weakening) and Lemma D.5 (Applying a context to a ground type)
☞ $\|\Theta\| \vdash [\Theta'][\Theta]\hat{\alpha} \cong^+ [\Theta']\hat{\alpha}$	By Lemma D.5 (Applying a context to a ground type)

- **Case** $\dfrac{\Theta \vdash N\,\text{type}^-}{\Theta \vdash {\downarrow}N\,\text{type}^+}$ Twfshift↓

$\Theta \vdash N\ \text{type}^-$	Subderivation
$\Theta \Longrightarrow \Theta'$	Assumption
$[\Theta']{\downarrow}N$ ground	Assumption
$[\Theta']N$ ground	By definition of ground
Θ ctx	Assumption
Θ' ctx	Assumption
$\|\Theta\| \vdash [\Theta'][\Theta]N \cong^- [\Theta']N$	By i.h.
☞ $\|\Theta\| \vdash [\Theta'][\Theta]{\downarrow}N \cong^+ [\Theta']{\downarrow}N$	By $\leq^{\pm}$Dshift↓ and definition of $[-]-$

- **Case** $\dfrac{\Theta, \alpha \vdash N\,\text{type}^-}{\Theta \vdash \forall\alpha.\,N\,\text{type}^-}$ Twfforall

$\Theta, \alpha \vdash N\ \text{type}^-$	Subderivation
$\Theta \Longrightarrow \Theta'$	Assumption
$\Theta, \alpha \Longrightarrow \Theta', \alpha$	By Wcuvar
$[\Theta']\forall\alpha.\,N$ ground	Assumption
$[\Theta', \alpha]N$ ground	By definition of ground
Θ ctx	Assumption

$\Theta, \alpha \ \mathsf{ctx}$	By Cwfuvar
$\Theta' \ \mathsf{ctx}$	Assumption
$\Theta', \alpha \ \mathsf{ctx}$	By Cwfuvar
$\|\Theta, \alpha\| \vdash [\Theta', \alpha][\Theta, \alpha]N \cong^- [\Theta', \alpha]N$	By i.h.
$\|\Theta, \alpha\| \vdash [\Theta'][\Theta]N \cong^- [\Theta']N$	By definition of $[-]-$
$\|\Theta\|, \alpha \vdash [\Theta'][\Theta]N \cong^- [\Theta']N$	By definition of $\|-\|$
$\|\Theta\|, \alpha \vdash \alpha \ \mathsf{type}^+$	By Twfuvar
$\|\Theta\|, \alpha \vdash [\Theta'][\Theta]\forall\alpha.\, N \leq^- [\Theta']N$	By $\leq^\pm$Dforalll and definition of $[-]-$
$\|\Theta\| \vdash [\Theta'][\Theta]\forall\alpha.\, N \leq^- [\Theta']\forall\alpha.\, N$	By $\leq^\pm$Dforallr ($\alpha \notin FUV(N)$) and definition of $[-]-$
$\|\Theta\|, \alpha \vdash \alpha \ \mathsf{type}^+$	Above
$\|\Theta\|, \alpha \vdash [\Theta']\forall\alpha.\, N \leq^- [\Theta'][\Theta]N$	By $\leq^\pm$Dforalll and definition of $[-]-$
$\|\Theta\| \vdash [\Theta']\forall\alpha.\, N \leq^- [\Theta'][\Theta]\forall\alpha.\, N$	By $\leq^\pm$Dforallr ($\alpha \notin FUV(N)$) and definition of $[-]-$
☞ $\|\Theta\| \vdash [\Theta'][\Theta]\forall\alpha.\, N \cong^- [\Theta']\forall\alpha.\, N$	Since we have the two component judgments

- **Case** $\dfrac{\Theta \vdash P \ \mathsf{type}^+ \qquad \Theta \vdash N \ \mathsf{type}^-}{\Theta \vdash P \to N \ \mathsf{type}^-}$ Twfarrow

$\Theta \vdash P \ \mathsf{type}^+$	Subderivation
$\Theta \Longrightarrow \Theta'$	Assumption
$[\Theta'](P \to N) \ \mathsf{ground}$	Assumption
$[\Theta']P \ \mathsf{ground}$	By definition of ground
$\Theta \ \mathsf{ctx}$	Assumption
$\Theta' \ \mathsf{ctx}$	Assumption
$\|\Theta\| \vdash [\Theta'][\Theta]P \cong^+ [\Theta']P$	By i.h.
$\Theta \vdash N \ \mathsf{type}^-$	Subderivation
$\Theta \Longrightarrow \Theta'$	Above
$[\Theta']N \ \mathsf{ground}$	By definition of ground
$\Theta \ \mathsf{ctx}$	Above
$\Theta' \ \mathsf{ctx}$	Above
$\|\Theta\| \vdash [\Theta'][\Theta]N \cong^- [\Theta']N$	By i.h.
☞ $\|\Theta\| \vdash [\Theta'][\Theta](P \to N) \cong^- [\Theta'](P \to N)$	By $\leq^\pm$Darrow and definition of $[-]-$

- **Case** $\dfrac{\Theta \vdash P \ \mathsf{type}^+}{\Theta \vdash \uparrow P \ \mathsf{type}^-}$ Twfshift$\uparrow$

  Symmetric to Twfshift$\downarrow$ case.

□

Lemma F.3 ($\Longrightarrow$ leads to isomorphic types (ground)). *If:*

1. $\Theta \vdash A \ \mathsf{type}^\pm$

2. $[\Theta]A$ *ground*

3. $\Theta \Longrightarrow \Theta'$

4. Θ ctx

5. Θ' ctx

then $\|\Theta\| \vdash [\Theta]A \cong^{\pm} [\Theta']A$.

Proof. Corollary of Lemma F.2.

	$\Theta \vdash A \ \text{type}^{\pm}$	Assumption
	$\Theta \Longrightarrow \Theta'$	"
	$[\Theta]A \ \text{ground}$	Assumption
	$[\Theta']A \ \text{ground}$	By Lemma C.8 (Extending context preserves groundness (weak))
	$\Theta \ \text{ctx}$	"
	$\Theta' \ \text{ctx}$	"
	$\|\Theta\| \vdash [\Theta'][\Theta]A \cong^{\pm} [\Theta']A$	By Lemma F.2 ($\Longrightarrow$ leads to isomorphic types)
☞	$\|\Theta\| \vdash [\Theta]A \cong^{\pm} [\Theta']A$	By Lemma D.5 (Applying a context to a ground type)

□

Lemma F.4 ($\longrightarrow$ leads to isomorphic types). *If:*

1. $\Theta \vdash A$ type$^{\pm}$

2. $\Theta \longrightarrow \Theta'$

3. $[\Theta']A$ *ground*

4. Θ ctx

5. Θ' ctx

then $\|\Theta\| \vdash [\Theta'][\Theta]A \cong^{\pm} [\Theta']A$.

Proof. Corollary of Lemma F.2 ($\Longrightarrow$ leads to isomorphic types).

	$\Theta \vdash A \ \text{type}^{\pm}$	Assumption
	$\Theta \longrightarrow \Theta'$	Assumption
	$\Theta \Longrightarrow \Theta'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
	$[\Theta']A \ \text{ground}$	Assumption
	$\Theta \ \text{ctx}$	Assumption
	$\Theta' \ \text{ctx}$	Assumption
☞	$\|\Theta\| \vdash [\Theta'][\Theta]A \cong^{\pm} [\Theta']A$	By Lemma F.2 ($\Longrightarrow$ leads to isomorphic types)

□

Lemma F.5 ($\longrightarrow$ leads to isomorphic types (ground)). *If:*

1. $\Theta \vdash A$ type$^{\pm}$

2. $[\Theta]A$ *ground*

3. $\Theta \longrightarrow \Theta'$

4. Θ ctx

5. Θ' ctx

then $\|\Theta\| \vdash [\Theta]A \cong^{\pm} [\Theta']A$.

Proof. Corollary of Lemma F.3 ($\Longrightarrow$ leads to isomorphic types (ground)).

	$\Theta \vdash A$ type$^{\pm}$	Assumption
	$[\Theta]A$ ground	Assumption
	$\Theta \longrightarrow \Theta'$	Assumption
	$\Theta \Longrightarrow \Theta'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
	Θ ctx	Assumption
	Θ' ctx	Assumption
☞	$\|\Theta\| \vdash [\Theta]A \cong^{\pm} [\Theta']A$	By Lemma F.3 ($\Longrightarrow$ leads to isomorphic types (ground))

□

F'.2 Statement

Theorem F.6 (Soundness of algorithmic subtyping)**.** *Given a well-formed algorithmic context* Θ *and a well-formed complete context* Ω*:*

- *If* $\Theta \vdash P \leq^{+} Q \dashv \Theta'$*,* $\Theta' \longrightarrow \Omega$*,* P *ground,* $[\Theta]Q = Q$*,* $\Theta \vdash P\ \text{type}^{+}$*, and* $\Theta \vdash Q\ \text{type}^{+}$*,*
  then $\|\Theta\| \vdash P \leq^{+} [\Omega]Q$.
- *If* $\Theta \vdash N \leq^{-} M \dashv \Theta'$*,* $\Theta' \longrightarrow \Omega$*,* M *ground,* $[\Theta]N = N$*,* $\Theta \vdash N\ \text{type}^{-}$*, and* $\Theta \vdash M\ \text{type}^{-}$*,*
  then $\|\Theta\| \vdash [\Omega]N \leq^{-} M$.

Proof. By mutual induction on the derivation of $\Theta \vdash P \leq^{\pm} Q \dashv \Theta'$.

- **Case**
$$\frac{}{\Theta_L, \alpha, \Theta_R \vdash \alpha \leq^{+} \alpha \dashv \Theta_L, \alpha, \Theta_R} \leq^{\pm}\text{Arefl}$$

	$\alpha \in UV(\|\Theta_L, \alpha, \Theta_R\|)$	By definition of $[-]-$
	$\|\Theta_L, \alpha, \Theta_R\| \vdash \alpha$ type^{+}	By Twfuvar
	$\|\Theta_L, \alpha, \Theta_R\| \vdash \alpha \leq^{+} \alpha$	By $\leq^{\pm}$Drefl
☞	$\|\Theta_L, \alpha, \Theta_R\| \vdash \alpha \leq^{+} [\Omega]\alpha$	By Lemma D.5 (Applying a context to a ground type)

- **Case**
$$\frac{\Theta_L \vdash P\ \text{type}^{+} \qquad P\ \text{ground}}{\Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^{+} \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R} \leq^{\pm}\text{Ainst}$$

$\Theta_L, \hat{\alpha} = P, \Theta_R \longrightarrow \Omega$	Assumption
$\Omega = \Omega_L, \hat{\alpha} = Q, \Omega_R$	Inversion (must have instance of Csolvedguess)

$$\begin{array}{rl}
\|\Theta_L\| \vdash P \cong^+ Q & '' \\
[\Omega]\hat{\alpha} = [\Omega_L, \hat{\alpha} = Q, \Omega_R]\hat{\alpha} & \text{Substituting for } \Omega \\
= Q & \text{By definition of } [-]- \\
\|\Theta_L, \hat{\alpha}, \Theta_R\| \vdash P \leq^+ Q & \text{By Lemma A.4 (Declarative subtyping weakening)} \\
\|\Omega\| \vdash P \leq^+ [\Omega]\hat{\alpha} & \text{Substituting using above equations}
\end{array}$$

- **Case** $\dfrac{\Theta \vdash M \leq^- N \dashv \Theta' \qquad \Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''}{\Theta \vdash \downarrow N \leq^+ \downarrow M \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}\downarrow$

$$\begin{array}{rl}
\downarrow N \ \text{ground} & \text{Assumption} \\
[\Theta]\downarrow M = \downarrow M & ''
\end{array}$$

We have:

$$\begin{array}{rl}
\Theta \vdash M \leq^- N \dashv \Theta' & \text{Subderivation} \\
\Theta \ \text{ctx} & \text{Assumption} \\
N \ \text{ground} & \text{By definition of ground} \\
[\Theta]M = M & \text{By definition of } [-]-
\end{array}$$

Therefore:

$$\begin{array}{rl}
\Theta' \ \text{ctx} & \text{By Lemma E.2 (Algorithmic subtyping is w.f.)} \\
\Theta \longrightarrow \Theta' & '' \\
[\Theta']M \ \text{ground} & ''
\end{array}$$

We have:

$$\begin{array}{rl}
\Theta' \vdash N \leq^- [\Theta']M \dashv \Theta'' & \text{Subderivation} \\
\Theta' \ \text{ctx} & \text{Above} \\
[\Theta']M \ \text{ground} & \text{Above} \\
[\Theta']N = N & \text{By Lemma D.5 (Applying a context to a ground type)}
\end{array}$$

Therefore:

$$\begin{array}{rl}
\Theta' \longrightarrow \Theta'' & \text{By Lemma E.2 (Algorithmic subtyping is w.f.)}
\end{array}$$

Now show the antecedents of the induction hypothesis applied to the first premise of the algorithmic rule:

$$\begin{array}{lrl}
(1) & \Theta \ \text{ctx} & \text{Above} \\
(2) & \Omega \ \text{ctx} & \text{Assumption} \\
(3) & \Theta \vdash M \leq^- N \dashv \Theta' & \text{Subderivation} \\
 & \Theta'' \longrightarrow \Omega & \text{Assumption} \\
(4) & \Theta' \longrightarrow \Omega & \text{By Lemma D.3 (Context extension is transitive)} \\
(5) & N \ \text{ground} & \text{Above} \\
(6) & [\Theta]M = M & \text{Above} \\
(7) & \Theta \vdash N \ \text{type}^- & \text{Inversion on assumption (Twfshift}\downarrow) \\
(8) & \Theta \vdash M \ \text{type}^- & ''
\end{array}$$

We conclude:

$$\begin{array}{rl}
\|\Theta\| \vdash [\Omega]M \leq^- N & \text{By i.h. applied to first premise, using (1–8)}
\end{array}$$

Show the antecedents of the induction hypothesis applied this time to the second premise of the algorithmic rule:

(9)	Θ' ctx	Above
(10)	Ω ctx	Above
(11)	$\Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''$	Subderivation
(12)	$\Theta'' \longrightarrow \Omega$	Assumption
(13)	$[\Theta']M$ ground	Above
(14)	$[\Theta']N = N$	Above
(15)	$\Theta' \vdash N$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)
(16)	$\Theta' \vdash M$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)

We conclude:

$\|\Theta'\| \vdash [\Omega]N \leq^- [\Theta']M$	By i.h. applied to second premise, using (9–16)
$\|\Theta'\| \vdash N \leq^- [\Theta']M$	By Lemma D.5 (Applying a context to a ground type)
$\leq^- [\Omega]M$	By Lemma F.5 ($\longrightarrow$ leads to isomorphic types (ground))
$\|\Theta\| \vdash N \leq^- [\Omega]M$	By Lemma D.2 (Equality of declarative contexts)

Applying the declarative rule:

	$\|\Theta\| \vdash {\downarrow}N \leq^+ {\downarrow}[\Omega]M$	By $\leq^\pm$Dshift$\downarrow$
☞	$\|\Theta\| \vdash {\downarrow}N \leq^+ [\Omega]{\downarrow}M$	By definition of $[-]-$

- **Case** $$\frac{\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha}{\Theta \vdash N \leq^- \forall\alpha.\, M \dashv \Theta'} \leq^\pm\text{Aforallr}$$

	Θ ctx	Assumption
(1)	Θ, α ctx	By Cwfuvar
	Ω ctx	Assumption
(2)	Ω, α ctx	By Cwfuvar
(3)	$\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha$	Subderivation
	$\Theta' \longrightarrow \Omega$	Assumption
(4)	$\Theta', \alpha \longrightarrow \Omega, \alpha$	By Cuvar
	$\forall\alpha.\, M$ ground	Assumption
(5)	M ground	Definition of ground
(6)	$[\Theta]N = N$	Assumption
	$\Theta \vdash N$ type$^-$	Assumption
(7)	$\Theta, \alpha \vdash N$ type$^-$	By Lemma A.2 (Term well-formedness weakening)
	$\Theta \vdash \forall\alpha.\, M$ type$^-$	Assumption
(8)	$\Theta, \alpha \vdash M$ type$^-$	Inversion (Twfforall)

	$\|\Theta, \alpha\| \vdash [\Omega, \alpha]N \leq^- M$	By i.h., using (1–8)
	$\|\Theta, \alpha\| \vdash [\Omega]N \leq^- M$	By definition of $[-]-$
☞	$\|\Theta\| \vdash [\Omega]N \leq^- \forall\alpha.\, M$	By $\leq^\pm$Dforallr

- **Case** $\dfrac{\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^- M \dashv \Theta', \hat{\alpha}[=P] \qquad M \neq \forall \alpha.\, M'}{\Theta \vdash \forall \alpha.\, N \leq^- M \dashv \Theta'} \leq^{\pm}\text{AforallI}$

Apply well-formedness to the premise:

(1)	$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^- M \dashv \Theta', \hat{\alpha}[=P]$	Subderivation
	Θ ctx	Assumption
(2)	$\Theta, \hat{\alpha}$ ctx	By Cwfunsolvedguess
(3)	M ground	Assumption
	$[\Theta]\forall\alpha.\, N = \forall\alpha.\, N$	Assumption
(4)	$[\Theta]N = N$	By definition of $[-]-$
	$\Theta', \hat{\alpha}[=P]$ ctx	By Lemma E.2 (Algorithmic subtyping is w.f.), using (1–4)
	$\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha}[=P]$	"

Now apply the inductive hypothesis:

(5)	$\Theta, \hat{\alpha}$ ctx	Above
	Ω ctx	Assumption
	$\Theta' \vdash P$ type$^+$	Inversion Cwfsolvedguess)
	P ground	"
	$\Theta' \longrightarrow \Omega$	Assumption
	$\Omega \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
(6)	$\Omega, \hat{\alpha} = P$ ctx	By Cwfsolvedguess
(7)	$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^- M \dashv \Theta', \hat{\alpha}[=P]$	Subderivation
	$\Theta' \longrightarrow \Omega$	Assumption
(8)	$\Theta', \hat{\alpha}[=P] \longrightarrow \Omega, \hat{\alpha} = P$	By Csolveguess/ Csolvedguess
(9)	M ground	Assumption
	$[\Theta]\forall\alpha.\, N = \forall\alpha.\, N$	Assumption
	$[\Theta]N = N$	By definition of $[-]-$
(10)	$[\Theta, \hat{\alpha}][\hat{\alpha}/\alpha]N = [\hat{\alpha}/\alpha]N$	$\Theta, \hat{\alpha}$ ctx, so $\hat{\alpha} \notin EV(\Theta)$
(11)	$\Theta \vdash \forall\alpha.\, N$ type$^-$	Assumption
(12)	$\Theta, \alpha \vdash N$ type$^-$	Inversion (Twfforall)
	$\|\Theta, \hat{\alpha}\| \vdash [\Omega, \hat{\alpha} = P][\hat{\alpha}/\alpha]N \leq^- M$	By i.h., using (5–12)
	$\|\Theta\| \vdash [\Omega, \hat{\alpha} = P][\hat{\alpha}/\alpha]N \leq^- M$	By definition of $\|-\|$
	$\alpha \notin UV(\Theta')$	Since $\alpha \notin UV(\Theta)$ as α fresh
	$[\Omega, \hat{\alpha} = P][\hat{\alpha}/\alpha]N = [\Omega][P/\hat{\alpha}][\hat{\alpha}/\alpha]N$	By definition of $[-]-$
	$= [\Omega][P/\hat{\alpha}][P/\alpha]N$	By composition of substitutions
	$= [\Omega][P/\alpha][P/\hat{\alpha}]N$	$\Theta' \vdash P$ type$^+$ and $\alpha \notin UV(\Theta')$, so $\alpha \notin FUV(P)$. Also P ground, so $\hat{\alpha} \notin FEV(P)$.
	$= [\Omega][P/\alpha]N$	Since $\hat{\alpha}$ fresh, $\hat{\alpha} \notin FEV(N)$
	$= [([\Omega]P)/\alpha][\Omega]N$	Since context application does not replace universal variables
	$\|\Theta\| \vdash [([\Omega]P)/\alpha][\Omega]N \leq^- M$	Substituting for $[\Omega, \hat{\alpha} = P][\hat{\alpha}/\alpha]N$
	$\Theta' \vdash P$ type$^+$	Above

$$
\begin{array}{rll}
P \;\; \text{ground} & & \text{Above} \\
\|\Theta'\| \vdash P \;\; \text{type}^+ & & \text{By Lemma F.1 (Completing context preserves w.f.)} \\
\|\Theta\| \vdash P \;\; \text{type}^+ & & \text{By Lemma D.2 (Equality of declarative contexts)} \\
\|\Theta\| \vdash [\Omega]P \;\; \text{type}^+ & & \text{By Lemma D.5 (Applying a context to a ground type)} \\
\|\Theta\| \vdash \forall\alpha.\, [\Omega]N \leq^- M & & \text{By } \leq^\pm\text{DforallI} \\
☞ \quad \|\Theta\| \vdash [\Omega]\forall\alpha.\, N \leq^- M & & \text{By definition of } [-]-
\end{array}
$$

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \dashv \Theta' \qquad \Theta' \vdash [\Theta']N \leq^- M \dashv \Theta''}{\Theta \vdash P \to N \leq^- Q \to M \dashv \Theta''} \leq^\pm\text{Aarrow}$

$$
\begin{array}{rl}
Q \to M \;\; \text{ground} & \text{Assumption} \\
[\Theta](P \to N) = P \to N & \text{Assumption} \\
\Theta \vdash P \to N \;\; \text{type}^- & \text{Assumption} \\
\Theta \vdash Q \to M \;\; \text{type}^- & \text{Assumption}
\end{array}
$$

We have:

$$
\begin{array}{rl}
\Theta \vdash Q \leq^+ P \dashv \Theta' & \text{Subderivation} \\
\Theta \;\; \text{ctx} & \text{Assumption} \\
Q \;\; \text{ground} & \text{Since } Q \to M \text{ ground} \\
[\Theta]P = P & \text{By definition of } [-]-
\end{array}
$$

Therefore by well-formedness:

$$
\begin{array}{rl}
\Theta' \;\; \text{ctx} & \text{By Lemma E.2 (Algorithmic subtyping is w.f.)} \\
\Theta \longrightarrow \Theta' & ''
\end{array}
$$

We have:

$$
\begin{array}{rl}
\Theta' \vdash [\Theta']N \leq^- M \dashv \Theta'' & \text{Subderivation} \\
\Theta' \;\; \text{ctx} & \text{Above} \\
M \;\; \text{ground} & \text{Since } Q \to M \text{ ground} \\
[\Theta'][\Theta']N = [\Theta']N & \text{By Lemma D.6 (Context application is idempotent)}
\end{array}
$$

Therefore by well-formedness:

$$
\begin{array}{rl}
\Theta' \longrightarrow \Theta'' & \text{By Lemma E.2 (Algorithmic subtyping is w.f.)}
\end{array}
$$

Applying the induction hypothesis to the first premise:

$$
\begin{array}{rl}
\Theta'' \longrightarrow \Omega & \text{Assumption} \\
\Theta' \longrightarrow \Omega & \text{By Lemma D.3 (Context extension is transitive)} \\
\Theta \vdash Q \;\; \text{type}^+ & \text{Inversion (Twfarrow)} \\
\Theta \vdash P \;\; \text{type}^+ & '' \\
\Omega \;\; \text{ctx} & \text{Assumption} \\
\|\Theta\| \vdash Q \leq^+ [\Omega]P & \text{By i.h. applied to first premise}
\end{array}
$$

Applying the induction hypothesis to the second premise:

$$
\begin{array}{rl}
\Theta'' \longrightarrow \Omega & \text{Assumption}
\end{array}
$$

$$\begin{array}{rl}
\Theta \vdash N \ \text{type}^- & \text{Inversion (Twfarrow)} \\
\Theta \vdash M \ \text{type}^- & '' \\
\Omega \ \text{ctx} & \text{Assumption} \\
\|\Theta'\| \vdash [\Omega][\Theta']N \leq^- M & \text{By i.h. applied to second premise}
\end{array}$$

Rework the second declarative judgment:

$$\begin{array}{rl}
\Theta' \longrightarrow \Omega & \text{Above} \\
\|\Theta'\| \vdash [\Omega]N \leq^- [\Omega][\Theta']N & \text{By Lemma F.4 (} \longrightarrow \text{ leads to isomorphic types)} \\
\leq^- M & \text{By Lemma B.7 (Declarative subtyping is transitive)} \\
\|\Theta\| \vdash [\Omega]N \leq^- M & \text{By Lemma D.2 (Equality of declarative contexts)}
\end{array}$$

Finally, apply the declarative rule:

$$\begin{array}{rl}
\|\Theta\| \vdash [\Omega]P \to [\Omega]N \leq^- Q \to M & \text{By } \leq^{\pm}\text{Darrow} \\
☞ \quad \|\Theta\| \vdash [\Omega](P \to N) \leq^- Q \to M & \text{By definition of } [-]-
\end{array}$$

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \dashv \Theta' \qquad \Theta' \vdash [\Theta']P \leq^+ Q \dashv \Theta''}{\Theta \vdash \uparrow P \leq^- \uparrow Q \dashv \Theta''} \leq^{\pm}\text{Ashift}\uparrow$

  Symmetric to $\leq^{\pm}$Ashift$\downarrow$ case.

□

G' Completeness of subtyping

G'.1 Lemmas for completeness

Lemma G.1 (Completion preserves w.f.). *If* Θ ctx, $\Theta \vdash A$ type$^{\pm}$*, and* $\Theta \longrightarrow \Omega$*, then* $\|\Theta\| \vdash [\Omega]A$ type$^{\pm}$.

Proof. Corollary of Lemma F.1 (Completing context preserves w.f.).

By $\Theta \vdash A$ type$^{\pm}$, all existential variables in A will appear in Θ. By $\Theta \longrightarrow \Omega$, these will also all appear in Ω as ground types. Therefore $[\Omega]A$ must be ground. Then:

$$\begin{array}{rl}
\Theta \vdash A \ \text{type}^{\pm} & \text{Assumption} \\
\Theta \vdash [\Omega]A \ \text{type}^{\pm} & \text{By Lemma E.1 (Applying context to the type preserves w.f.)} \\
[\Omega]A \ \text{ground} & \text{Above} \\
\|\Theta\| \vdash [\Omega]A \ \text{type}^{\pm} & \text{By Lemma F.1 (Completing context preserves w.f.)}
\end{array}$$

□

Lemma G.2 (Extension solving guess). *If* $\Theta_L, \hat{\alpha}, \Theta_R \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R$ *and* $[\Omega_L]\Theta_L \vdash P \cong^+ Q$*, then* $\Theta_L, \hat{\alpha} = P, \Theta_R \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R$.

Proof. By structural induction on Θ_R.

- **Case** $\Theta_R = \cdot$:

$\Theta_L, \hat{\alpha} \longrightarrow \Omega_L, \hat{\alpha} = Q$	Assumption
$\Theta_L \longrightarrow \Omega_L$	Inversion (Csolveguess)
$[\Omega_L]\Theta_L \vdash P \cong^+ Q$	Assumption
☞ $\Theta_L, \hat{\alpha} = P \longrightarrow \Omega_L, \hat{\alpha} = Q$	By Csolvedguess

- **Case** $\Theta_R = \Theta_R', \alpha$:

$\Theta_L, \hat{\alpha}, \Theta_R', \alpha \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R', \alpha$	By structure of Θ_R, must have instance of Cuvar
$[\Omega_L]\Theta_L \vdash P \cong^+ Q$	Assumption
$\Theta_L, \hat{\alpha}, \Theta_R' \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R'$	Inversion (Cuvar)
$\Theta_L, \hat{\alpha} = P, \Theta_R' \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R'$	By i.h.
☞ $\Theta_L, \hat{\alpha} = P, \Theta_R', \alpha \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R', \alpha$	By Cuvar

- **Case** $\Theta_R = \Theta_R', \hat{\beta}$:

$\Theta_L, \hat{\alpha}, \Theta_R', \hat{\beta} \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R', \hat{\beta} = R$	By structure of Θ_R, must have instance of Csolveguess
$[\Omega_L]\Theta_L \vdash P \cong^+ Q$	Assumption
$\Theta_L, \hat{\alpha}, \Theta_R' \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R'$	Inversion (Csolveguess)
$\Theta_L, \hat{\alpha} = P, \Theta_R' \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R'$	By i.h.
☞ $\Theta_L, \hat{\alpha} = P, \Theta_R', \hat{\beta} \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R', \hat{\beta} = R$	By Csolveguess

- **Case** $\Theta_R = \Theta_R', \hat{\beta} = R$:

$\Theta_L, \hat{\alpha}, \Theta_R', \hat{\beta} = R \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R', \hat{\beta} = S$	By structure of Θ_R, must have instance of Csolvedguess
$[\Omega_L, \hat{\alpha} = Q, \Omega_R'](\Theta_L, \hat{\alpha}, \Theta_R') \vdash R \cong^+ S$	"
$[\Omega_L]\Theta_L \vdash P \cong^+ Q$	Assumption
$\Theta_L, \hat{\alpha}, \Theta_R' \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R'$	Inversion (Csolveguess)
$\Theta_L, \hat{\alpha} = P, \Theta_R' \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R'$	By i.h.
$[\Omega_L, \hat{\alpha} = Q, \Omega_R'](\Theta_L, \hat{\alpha} = P, \Theta_R') \vdash R \cong^+ S$	Since $[-]-$ ignores existential variables
☞ $\Theta_L, \hat{\alpha} = P, \Theta_R', \hat{\beta} = R \longrightarrow \Omega_L, \hat{\alpha} = Q, \Omega_R', \hat{\beta} = S$	By Csolvedguess

□

Lemma G.3 (Context extension substitution size)**.** *If:*

1. Θ ctx

2. $\Theta \vdash A\ \text{type}^{\pm}$

3. $\Theta \longrightarrow \Omega$

4. Ω ctx

then $|[\Omega][\Theta]A|_{NQ} = |[\Omega]A|_{NQ}$.

Proof. Corollary of Lemma F.4 ($\longrightarrow$ leads to isomorphic types) and Lemma B.6 (Isomorphic types are the same size).

	$\Theta \vdash A$ type$^{\pm}$	Assumption
	$[\Omega]A$ ground	Ω completes all free existential variables in A
	$\Theta \longrightarrow \Omega$	"
	$\|\Theta\| \vdash [\Omega][\Theta]A \cong^{\pm} [\Omega]A$	By Lemma F.4 ($\longrightarrow$ leads to isomorphic types)
	Θ ctx	Assumption
	$\Theta \vdash [\Theta]A$ type$^{\pm}$	By Lemma E.1 (Applying context to the type preserves w.f.)
	$\|\Theta\| \vdash [\Omega][\Theta]A$ type$^{\pm}$	By Lemma G.1 (Completion preserves w.f.)
	Ω ctx	Assumption
	$\|\Theta\| \vdash [\Omega]A$ type$^{\pm}$	By Lemma G.1 (Completion preserves w.f.)
☞	$\|[\Omega][\Theta]A\|_{NQ} = \|[\Omega]A\|_{NQ}$	By Lemma B.6 (Isomorphic types are the same size)

□

Lemma G.4 (Context extension ground substitution size)**.** *If:*

1. Θ ctx

2. $\Theta \vdash A$ type$^{\pm}$

3. $[\Theta]A$ *ground*

4. $\Theta \longrightarrow \Omega$

5. Ω ctx

then $|[\Theta]A|_{NQ} = |[\Omega]A|_{NQ}$.

Proof. Corollary of Lemma G.4 (Context extension ground substitution size).

	$\|[\Omega][\Theta]A\|_{NQ} = \|[\Omega]A\|_{NQ}$	By Lemma B.6 (Isomorphic types are the same size)
☞	$\|[\Theta]A\|_{NQ} = \|[\Omega]A\|_{NQ}$	By Lemma D.5 (Applying a context to a ground type)

□

G'.2 Statement

Theorem G.5 (Completeness of algorithmic subtyping)**.** *If* Θ ctx, $\Theta \longrightarrow \Omega$*, and* Ω ctx*, then:*

- *If* $\|\Theta\| \vdash P \leq^{+} [\Omega]Q$, $\Theta \vdash P$ type^{+}, $\Theta \vdash Q$ type^{+}, P *ground, and* $[\Theta]Q = Q$*, then* $\exists\Theta'$ *such that* $\Theta \vdash P \leq^{+} Q \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.

- *If* $\|\Theta\| \vdash [\Omega]N \leq^{-} M$, $\Theta \vdash M$ type^{-}, $\Theta \vdash N$ type^{-}, M *ground, and* $[\Theta]N = N$*, then* $\exists\Theta'$ *such that* $\Theta \vdash N \leq^{-} M \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.

Proof. By mutual rule induction on the declarative judgment weighted by the lexicographic ordering of $(|P|_{NQ}, \mathrm{NPQ}(P) + \mathrm{NPQ}(Q))$ in the positive case where we have $\|\Theta\| \vdash P \leq^+ [\Omega]Q$, and $(|M|_{NQ}, \mathrm{NPQ}(N) + \mathrm{NPQ}(M))$ in the negative case where we have $\|\Theta\| \vdash [\Omega]N \leq^- M$.

Firstly consider the case where $B = \hat{\alpha}$. Suppose $[\Omega]\hat{\alpha} = Q'$:

	$\Theta = \Theta_L, \hat{\alpha}, \Theta_R$	Since $\Theta \vdash \hat{\alpha}$ type$^+$ and $[\Theta]\hat{\alpha} = \hat{\alpha}$ by assumption
	$\Omega = \Omega_L, \hat{\alpha} = Q', \Omega_R$	Since $[\Omega]\hat{\alpha} = Q'$
	$\Omega_L \vdash Q'$ type$^+$	Inversion on Ω ctx (Cwfsolvedguess)
	Q' ground	"
	$\Theta_L, \hat{\alpha}, \Theta_R \longrightarrow \Omega_L, \hat{\alpha} = Q', \Omega_R$	Assumption
	$\Theta_L \longrightarrow \Omega_L$	Inversion (must have instance of Csolveguess)
	$\Theta_L \vdash Q'$ type$^+$	By the rules defining $\longrightarrow$, each uvar in Ω_L must appear in Θ_L
	$\|\Theta\| \vdash P \leq^+ Q'$	Assumption
	P ground	"
	$\mathrm{FUV}(P) \subseteq \mathrm{UV}(\Theta_L)$	Since $\|\Theta\| \vdash P \leq^+ Q'$ and $\Theta_L \vdash Q'$ type$^+$
	$\mathrm{FEV}(P) \subseteq \mathrm{EV}(\Theta_L)$	$\mathrm{FEV}(P) = \varnothing$ since P ground
	$\Theta \vdash P$ type$^+$	Assumption
	$\Theta_L \vdash P$ type$^+$	By above three equations
☞	$\Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^+ \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R$	By $\leq^\pm$Ainst
	$\|\Theta\| \vdash Q' \leq^+ P$	By Lemma B.4 (Symmetry of positive declarative subtyping)
	$\|\Theta\| \vdash P \cong^+ Q'$	Since we have both the component judgments
	$[\Omega]\Theta_L \vdash P \cong^+ Q'$	Since $\|\Theta\| \vdash P \cong^+ Q'$, $\Theta_L \vdash P$ type$^+$, and $\Theta_L \vdash Q'$ type$^+$
	$[\Omega_L]\Theta_L \vdash P \cong^+ Q'$	Since $\Theta_L \longrightarrow \Omega_L$, $\Theta_L \vdash P$ type$^+$, and $\Theta_L \vdash Q'$ type$^+$
	$\Theta_L, \hat{\alpha}, \Theta_R \longrightarrow \Omega_L, \hat{\alpha} = Q', \Omega_R$	Above
☞	$\Theta_L, \hat{\alpha} = P, \Theta_R \longrightarrow \Omega_L, \hat{\alpha} = Q', \Omega_R$	By Lemma G.2 (Extension solving guess)

Now consider the cases where $B \neq \hat{\alpha}$:

- **Case**

$$\frac{\|\Theta\| \vdash \alpha \text{ type}^+}{\|\Theta\| \vdash \alpha \leq^+ [\Omega]\alpha} \leq^\pm\text{Drefl}$$

	$\|\Theta\| \vdash \alpha$ type$^+$	Subderivation
	$\alpha \in \mathrm{UV}([\Omega]\Theta)$	Inversion (Twfuvar)
	$\alpha \in \mathrm{UV}(\Theta)$	By definition of $[-]-$
	$\Theta = \Theta_L, \alpha, \Theta_R$	Since $\alpha \in \mathrm{UV}(\Theta)$
☞	$\Theta_L, \alpha, \Theta_R \vdash \alpha \leq^+ \alpha \dashv \Theta_L, \alpha, \Theta_R$	By $\leq^\pm$Arefl
☞	$\Theta_L, \alpha, \Theta_R \longrightarrow \Omega$	Assumption

- **Case**

$$\frac{\|\Theta\| \vdash [\Omega]M \leq^- N \qquad \|\Theta\| \vdash N \leq^- [\Omega]M}{\|\Theta\| \vdash {\downarrow}N \leq^+ [\Omega]{\downarrow}M} \leq^\pm\text{Dshift}{\downarrow}$$

$\Theta \vdash {\downarrow}N$ type$^+$	Assumption
$\Theta \vdash {\downarrow}M$ type$^+$	Assumption

	$\downarrow N$ ground	Assumption
	$[\Theta]\downarrow M = \downarrow M$	Assumption
	$\lVert\Theta\rVert \vdash [\Omega]M \leq^- N$	Subderivation
	Θ ctx	Assumption
	$\Theta \vdash M$ type$^-$	Inversion (Twfshift$\downarrow$)
	$\Theta \vdash N$ type$^-$	"
	$\Theta \longrightarrow \Omega$	Assumption
	Ω ctx	"
	N ground	By definition of ground
	$[\Theta]M = M$	By definition of $[-]-$
	$\Theta \vdash M \leq^- N \dashv \Theta'$	By i.h. (the type size of the ground side type in the declarative judgment has decreased)
	$\Theta' \longrightarrow \Omega$	"
	Θ' ctx	By Lemma E.2 (Algorithmic subtyping is w.f.)
	$\Theta \longrightarrow \Theta'$	"
	$[\Theta']M$ ground	"
	$\lVert\Theta\rVert \vdash N \leq^- [\Omega]M$	Subderivation
	$\lVert\Theta'\rVert \vdash N \leq^- [\Omega]M$	By Lemma D.2 (Equality of declarative contexts)
	$\lVert\Theta'\rVert \vdash [\Theta']M \cong^- [\Omega]M$	By Lemma F.5 ($\longrightarrow$ leads to isomorphic types (ground))
	$\Theta' \vdash N$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)
	$\lVert\Theta'\rVert \vdash N$ type$^-$	By Lemma F.1 (Completing context preserves w.f.)
	$\Theta' \vdash M$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)
	$\lVert\Theta'\rVert \vdash [\Omega]M$ type$^-$	By Lemma G.1 (Completion preserves w.f.)
	$\Theta' \vdash [\Theta']M$ type$^-$	By Lemma E.1 (Applying context to the type preserves w.f.)
	$\lVert\Theta'\rVert \vdash [\Theta']M$ type$^-$	By Lemma F.1 (Completing context preserves w.f.)
	$\lVert\Theta'\rVert \vdash N \leq^- [\Theta']M$	By Lemma B.7 (Declarative subtyping is transitive)
	$\lVert\Theta'\rVert \vdash [\Omega]N \leq^- [\Theta']M$	By Lemma D.5 (Applying a context to a ground type)
	Θ' ctx	Above
	$\Theta' \vdash N$ type$^-$	Above
	$\Theta' \vdash [\Theta']M$ type$^-$	Above
	$\Theta' \longrightarrow \Omega$	Above
	Ω ctx	Above
	$[\Theta']M$ ground	Above
	$[\Theta']N = N$	By Lemma D.5 (Applying a context to a ground type)
	$\lvert[\Theta']M\rvert_{NQ} = \lvert[\Omega]M\rvert_{NQ}$	By Lemma G.4 (Context extension ground substitution size)
	$= \lvert N\rvert_{NQ}$	By Lemma B.6 (Isomorphic types are the same size)
	$< \lvert\downarrow N\rvert_{NQ}$	By definition of $\lvert-\rvert_{NQ}$
	$\Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''$	By i.h. (the type size of the ground side type in the declarative judgment has decreased)
☞	$\Theta'' \longrightarrow \Omega$	"
☞	$\Theta \vdash \downarrow N \leq^+ \downarrow M \dashv \Theta''$	By $\leq^{\pm}$Ashift$\downarrow$

- **Case** $\dfrac{\|\Theta, \alpha\| \vdash [\Omega]N \leq^- M}{\|\Theta\| \vdash [\Omega]N \leq^- \forall\alpha. M} \leq^\pm\mathsf{Dforallr}$

Θ ctx	Assumption
$\Theta \vdash N$ type$^-$	Assumption
$\Theta \vdash \forall\alpha. M$ type$^-$	Assumption
$\Theta \longrightarrow \Omega$	Assumption
Ω ctx	Assumption
$\forall\alpha. M$ ground	Assumption
$[\Theta]N = N$	Assumption
$\|\Theta, \alpha\| \vdash [\Omega]N \leq^- M$	Subderivation
$\|\Theta, \alpha\| \vdash [\Omega, \alpha]N \leq^- M$	By definition of $[-]-$
Θ, α ctx	By Cwfuvar
$\Theta, \alpha \vdash N$ type$^-$	By Lemma A.2 (Term well-formedness weakening)
$\Theta, \alpha \vdash M$ type$^-$	Inversion (Twfforall)
$\Theta, \alpha \longrightarrow \Omega, \alpha$	By Cuvar
Ω, α ctx	By Cwfuvar
M ground	By definition of ground
$[\Theta, \alpha]N = N$	By definition of $[-]-$
$\Theta, \alpha \vdash N \leq^- M \dashv \Theta''$	By i.h. (the type size of the ground side type in the declarative judgment is the same and the total number of prenex quantifiers has decreased by 1)
$\Theta'' \longrightarrow \Omega, \alpha$	$''$
$\Theta'' = \Theta', \alpha$	Inversion (Cuvar)
☞ $\Theta' \longrightarrow \Omega$	$''$
$\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha$	Substituting for Θ''
☞ $\Theta \vdash N \leq^- \forall\alpha. M \dashv \Theta'$	By $\leq^\pm\mathsf{Aforallr}$

- **Case** $\dfrac{\|\Theta\| \vdash P\ \mathsf{type}^+ \qquad \|\Theta\| \vdash [\Omega][P/\alpha]N \leq^- M'}{\|\Theta\| \vdash [\Omega]\forall\alpha. N \leq^- M'} \leq^\pm\mathsf{Dforalll}$

Proof by induction on the number of prenex universal quantifiers in M':

  - **Case** $n = 0$ (base case). Let $M = M'$:

Θ ctx	Assumption
$\Theta \longrightarrow \Omega$	Assumption
Ω ctx	Assumption
$\|\Theta\| \vdash P$ type$^+$	Subderivation
$\Theta \vdash P$ type$^+$	Since P ground, this reduces to $FUV(P) \subseteq UV(\Theta)$, which holds since $[-]-$ preserves uvars and $FUV(P) \subseteq [\Omega]\Theta$ (the latter holding by $\|\Theta\| \vdash P\ \mathsf{type}^+$).
$\Theta \vdash \forall\alpha. N$ type$^-$	Assumption

$\Theta, \alpha \vdash N \ \mathsf{type}^-$	Inversion (Twfforall)
$\Theta \vdash M \ \mathsf{type}^-$	Assumption
$[\Theta]\forall\alpha.\, N = \forall\alpha.\, N$	Assumption
$[\Theta]N = N$	By definition of $[-]-$
$\|\Theta\| \vdash [\Omega][P/\alpha]N \leq^- M$	Subderivation
$[\Omega, \hat{\alpha} = P]\Theta, \hat{\alpha} \vdash$	
$[\Omega, \hat{\alpha} = P][\hat{\alpha}/\alpha]N \leq^- M$	Where $\hat{\alpha}$ is fresh
$\Theta, \hat{\alpha} \ \mathsf{ctx}$	By Cwfunsolvedguess
$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \ \mathsf{type}^-$	Each application of Twfuvar involving α becomes an application of Twfguess involving $\hat{\alpha}$, and $\hat{\alpha} \in \mathrm{EV}(\Theta, \hat{\alpha})$
$\Theta, \hat{\alpha} \vdash M \ \mathsf{type}^-$	By Lemma A.2 (Term well-formedness weakening)
$\Theta, \hat{\alpha} \longrightarrow \Omega, \hat{\alpha} = P$	By Csolveguess
$\Omega, \hat{\alpha} = P \ \mathsf{ctx}$	By Cwfsolvedguess and Lemma D.4 (Context extension preserves w.f.)
$M \ \mathsf{ground}$	Assumption
$[\Theta, \hat{\alpha}][\hat{\alpha}/\alpha]N = [\hat{\alpha}/\alpha]N$	$\Theta, \hat{\alpha}$ can not solve $\hat{\alpha}$ since $\Theta, \hat{\alpha}\,\mathsf{ctx}$, and $[\Theta]N = N$
$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^- M \dashv \Theta''$	By completeness i.h. (the type size of the ground side type in the declarative judgment is the same, but the total number of prenex quantifiers has decreased by 1)
$\Theta'' \longrightarrow \Omega, \hat{\alpha} = P$	″

By inversion on $\Theta'' \longrightarrow \Omega, \hat{\alpha} = P$, have $\Theta' \longrightarrow \Omega$ and one of the following cases:

Case $\Theta'' = \Theta', \hat{\alpha} = Q$ and $\|\Theta'\| \vdash Q \cong^+ P$:

	$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^- M \dashv \Theta', \hat{\alpha} = Q$	Substituting for Θ''
☞	$\Theta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta'$	By $\leq^\pm$Aforalll
☞	$\Theta' \longrightarrow \Omega$	Above

Case $\Theta'' = \Theta', \hat{\alpha}$ and $\alpha \notin \mathrm{FUV}(N)$:

	$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N \leq^- M \dashv \Theta', \hat{\alpha} = \times$	Where $\times$ represents "not solved"
☞	$\Theta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta'$	By $\leq^\pm$Aforalll
☞	$\Theta' \longrightarrow \Omega$	Above

- **Case** M' has $n+1$ prenex universal quantifiers, i.e. $M' = \forall\beta.\, M$ where M has n prenex universal quantifiers:

$\|\Theta, \beta\| \vdash [\Omega]\forall\alpha.\, N \leq^- M$	Inversion ($\leq^\pm$Dforallr)
$\|\Theta, \beta\| \vdash [\Omega, \beta]\forall\alpha.\, N \leq^- M$	By definition of $[-]-$
$\Theta, \beta \ \mathsf{ctx}$	By Cwfuvar
$\Theta, \beta \vdash \forall\alpha.\, N \ \mathsf{type}^-$	By Lemma A.2 (Term well-formedness weakening)
$\Theta, \beta \vdash M \ \mathsf{type}^-$	Inversion (Twfforall)
$\Theta, \beta \longrightarrow \Omega, \beta$	By Cuvar
$\Omega, \beta \ \mathsf{ctx}$	By Cwfuvar
$M \ \mathsf{ground}$	By definition of ground

	$[\Theta, \beta]\forall\alpha.\, N = \forall\alpha.\, N$	By definition of $[-]-$
	$\Theta, \beta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta''$	By i.h. of induction over prenex universal quantifiers
	$\Theta'' \longrightarrow \Omega, \beta$	$''$
	$\Theta'' = \Theta', \beta$	Inversion (Cuvar)
☞	$\Theta' \longrightarrow \Omega$	$''$
	$\Theta, \beta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta', \beta$	Substituting for Θ''
☞	$\Theta \vdash \forall\alpha.\, N \leq^- \forall\beta.\, M \dashv \Theta'$	By $\leq^\pm$Aforallr

- **Case** $\dfrac{\|\Theta\| \vdash Q \leq^+ [\Omega]P \qquad \|\Theta\| \vdash [\Omega]N \leq^- M}{\|\Theta\| \vdash [\Omega](P \to N) \leq^- (Q \to M)} \leq^\pm\text{Darrow}$

$Q \to M$ ground	Assumption
$[\Theta](P \to N) = (P \to N)$	Assumption
$\Theta \vdash P \to N$ type$^+$	Assumption
$\Theta \vdash Q \to M$ type$^+$	Assumption
$\|\Theta\| \vdash Q \leq^+ [\Omega]P$	Subderivation
Θ ctx	Assumption
$\Theta \vdash Q$ type$^+$	Inversion (Twfarrow)
$\Theta \vdash P$ type$^+$	$''$
$\Theta \longrightarrow \Omega$	Assumption
Ω ctx	Assumption
Q ground	By definition of ground
$[\Theta]P = P$	By definition of $[-]-$
$\Theta \vdash Q \leq^+ P \dashv \Theta'$	By i.h. (the type size of the ground side type of the declarative judgment has decreased)
$\Theta' \longrightarrow \Omega$	$''$
$\Theta \longrightarrow \Theta'$	By Lemma D.2 (Equality of declarative contexts)
Θ' ctx	$''$
$\|\Theta\| \vdash [\Omega]N \leq^- M$	Subderivation
$\|\Theta'\| \vdash [\Omega]N \leq^- M$	By Lemma D.2 (Equality of declarative contexts)
$\Theta' \vdash N$ type$^-$	By inversion (Twfarrow) and Lemma D.4 (Context extension preserves w.f.)
$\|\Theta'\| \vdash [\Omega][\Theta']N \cong^- [\Omega]N$	By Lemma F.4 ($\longrightarrow$ leads to isomorphic types)
$\Theta' \vdash [\Theta']N$ type$^-$	By Lemma E.1 (Applying context to the type preserves w.f.)
$\|\Theta'\| \vdash [\Omega][\Theta']N$ type$^-$	By Lemma G.1 (Completion preserves w.f.)
$\|\Theta'\| \vdash [\Omega]N$ type$^-$	By Lemma G.1 (Completion preserves w.f.)
$\Theta \vdash M$ type$^-$	Inversion (Twfarrow)
$\Theta' \vdash M$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)
$\|\Theta'\| \vdash M$ type$^-$	By Lemma F.1 (Completing context preserves w.f.)
$\|\Theta'\| \vdash [\Omega][\Theta']N \leq^- M$	By Lemma B.7 (Declarative subtyping is transitive)
Θ' ctx	Above
$\Theta' \vdash [\Theta']N$ type$^-$	Above
$\Theta' \vdash M$ type$^-$	Above

	$\Theta' \longrightarrow \Omega$	Above
	Ω ctx	Above
	M ground	Since $Q \rightarrow M$ ground
	$[\Theta'][\Theta']N = [\Theta']N$	By Lemma D.6 (Context application is idempotent)
	$\Theta' \vdash [\Theta']N \leq^- M \dashv \Theta''$	By i.h. (the type size of the ground side type of the declarative judgment has decreased)
☞	$\Theta'' \longrightarrow \Omega$	$''$
☞	$\Theta \vdash P \rightarrow N \leq^- Q \rightarrow M \dashv \Theta''$	By $\leq^\pm$Aarrow

- **Case** $$\dfrac{\|\Theta\| \vdash Q \leq^+ [\Omega]P \qquad \|\Theta\| \vdash [\Omega]P \leq^+ Q}{\|\Theta\| \vdash [\Omega]{\uparrow}P \leq^- {\uparrow}Q} \leq^\pm\mathsf{Dshift}{\uparrow}$$

$\Theta \vdash {\uparrow}P$ type$^-$	Assumption
$\Theta \vdash {\uparrow}Q$ type$^-$	Assumption
${\uparrow}Q$ ground	Assumption
$[\Theta]{\uparrow}P = {\uparrow}P$	Assumption
$\|\Theta\| \vdash Q \leq^+ [\Omega]P$	Subderivation
Θ ctx	Assumption
$\Theta \vdash Q$ type$^+$	Since $\Theta \vdash {\uparrow}Q$ type$^+$
$\Theta \vdash P$ type$^+$	Since $\Theta \vdash {\uparrow}P$ type$^+$
$\Theta \longrightarrow \Omega$	Assumption
Ω ctx	Assumption
Q ground	By definition of ground
$[\Theta]P = P$	By definition of $[-]-$
$\Theta \vdash Q \leq^+ P \dashv \Theta'$	By i.h.
$\Theta' \longrightarrow \Omega$	$''$
Θ' ctx	By Lemma E.2 (Algorithmic subtyping is w.f.)
$\Theta \longrightarrow \Theta'$	$''$
$[\Theta']P$ ground	$''$
$\|\Theta\| \vdash [\Omega]P \leq^+ Q$	Subderivation
$\|\Theta'\| \vdash [\Omega]P \leq^+ Q$	By Lemma D.2 (Equality of declarative contexts)
$\Theta' \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
$\|\Theta'\| \vdash [\Theta']P \cong^+ [\Omega]P$	By Lemma F.5 ($\longrightarrow$ leads to isomorphic types (ground))
$\Theta' \vdash [\Theta']P$ type$^+$	By Lemma E.1 (Applying context to the type preserves w.f.)
$\|\Theta'\| \vdash [\Theta']P$ type$^+$	By Lemma F.1 (Completing context preserves w.f.)
$\|\Theta'\| \vdash [\Omega]P$ type$^+$	By Lemma G.1 (Completion preserves w.f.)
$\Theta' \vdash Q$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
$\|\Theta'\| \vdash Q$ type$^+$	By Lemma F.1 (Completing context preserves w.f.)
$\|\Theta'\| \vdash [\Theta']P \leq^+ Q$	By Lemma B.7 (Declarative subtyping is transitive)
$\|\Theta'\| \vdash [\Theta']P \leq^+ [\Omega]Q$	By Lemma D.5 (Applying a context to a ground type)
Θ' ctx	Above
$\Theta' \vdash [\Theta']P$ type$^+$	By Lemma E.1 (Applying context to the type preserves w.f.)
$\Theta' \vdash Q$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)

$$\begin{array}{lll}
& \Theta' \longrightarrow \Omega & \text{Above} \\
& \Omega \ \text{ctx} & \text{Above} \\
& [\Theta']P \ \text{ground} & \text{Above} \\
& [\Theta']Q = Q & \text{By Lemma D.5 (Applying a context to a ground type)} \\
& |[\Theta']P|_{NQ} = |[\Omega]P|_{NQ} & \text{By Lemma G.4 (Context extension ground substitution size)} \\
& = |[\Omega]Q|_{NQ} & \text{By Lemma B.6 (Isomorphic types are the same size)} \\
& < |[\Omega]{\uparrow}Q|_{NQ} & \text{By definition of } |-|_{NQ} \\
& \Theta' \vdash [\Theta']P \leq^{+} Q \dashv \Theta'' & \text{By i.h. (the type size of the ground side type of the declarative judgment has decreased)} \\
☞ & \Theta'' \longrightarrow \Omega & '' \\
☞ & \Theta \vdash {\uparrow}P \leq^{-} {\uparrow}Q \dashv \Theta'' & \text{By } \leq^{\pm}\text{Ashift}{\uparrow}
\end{array}$$

□

H' Determinism of subtyping

Lemma H.1 (Algorithmic subtyping is deterministic)**.**

- *If* $\Theta \vdash P \leq^{+} Q \dashv \Theta'_1$ *and* $\Theta \vdash P \leq^{+} Q \dashv \Theta'_2$*, then* $\Theta'_1 = \Theta'_2$.
- *If* $\Theta \vdash N \leq^{-} M \dashv \Theta'_1$ *and* $\Theta \vdash N \leq^{-} M \dashv \Theta'_2$*, then* $\Theta'_1 = \Theta'_2$.

Proof. By rule induction on the first hypothesis.

- **Case**

$$\frac{}{\Theta_L, \alpha, \Theta_R \vdash \alpha \leq^{+} \alpha \dashv \Theta_L, \alpha, \Theta_R} \leq^{\pm}\text{Arefl}$$

$$\begin{array}{lll}
& \Theta_L, \alpha, \Theta_R \vdash \alpha \leq^{+} \alpha \dashv \Theta_L, \alpha, \Theta_R & \text{Assumption} \\
& \Theta_L, \alpha, \Theta_R \vdash \alpha \leq^{+} \alpha \dashv \Theta'_2 & \text{Assumption} \\
☞ & \Theta_L, \alpha, \Theta_R = \Theta'_2 & \text{By the structure of } \alpha \text{, the instantiation above is the only possible instantiation of } \leq^{+}
\end{array}$$

- **Case**

$$\frac{\Theta' \vdash P \ \text{type}^{+} \qquad P \ \text{ground}}{\Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^{+} \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R} \leq^{\pm}\text{Ainst}$$

$$\begin{array}{lll}
& \Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^{+} \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R & \text{Assumption} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^{+} \hat{\alpha} \dashv \Theta'_2 & \text{Assumption} \\
☞ & \Theta_L, \hat{\alpha} = P, \Theta_R = \Theta'_2 & \text{By the structure of } \hat{\alpha} \text{, the instantiation above is the only possible instantiation of } \leq^{+}
\end{array}$$

- **Case** $\dfrac{\Theta \vdash M \leq^- N \dashv \Theta' \qquad \Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''}{\Theta \vdash {\downarrow}N \leq^+ {\downarrow}M \dashv \Theta''} \leq^\pm\mathsf{Ashift}{\downarrow}$

$$\begin{array}{ll} \Theta \vdash {\downarrow}N \leq^+ {\downarrow}M \dashv \Theta'' & \text{Assumption} \\ \Theta \vdash {\downarrow}N \leq^+ {\downarrow}M \dashv \Theta''_2 & \text{Assumption} \end{array}$$

By the structure of ${\downarrow}N$, the derivation of the second hypothesis must end with an application of the $\leq^\pm\mathsf{Ashift}{\downarrow}$ rule.

$$\begin{array}{ll} \Theta \vdash M \leq^- N \dashv \Theta' & \text{Subderivation} \\ \Theta \vdash M \leq^- N \dashv \Theta'_2 & \text{Subderivation} \\ \Theta' = \Theta'_2 & \text{By i.h.} \end{array}$$

$$\begin{array}{ll} \Theta' \vdash N \leq^- [\Theta']M \dashv \Theta'' & \text{Subderivation} \\ \Theta'_2 \vdash N \leq^- [\Theta'_2]M \dashv \Theta''_2 & \text{Subderivation} \\ \Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''_2 & \text{Using } \Theta' = \Theta'_2 \\ ☞ \quad \Theta'' = \Theta''_2 & \text{By i.h.} \end{array}$$

- **Case** $\dfrac{\Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N \leq^- M \dashv \Theta', \hat\alpha\,[= P] \qquad M \neq \forall\beta.\, M'}{\Theta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta'} \leq^\pm\mathsf{Aforalll}$

$$\begin{array}{ll} \Theta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta' & \text{Assumption} \\ \Theta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta'_2 & \text{Assumption} \end{array}$$

By the structure of $\forall\alpha.\, N$, and since $M \neq \forall\beta.\, M'$, the derivation of the second hypothesis must conclude with an application of $\leq^\pm\mathsf{Aforalll}$.

$$\begin{array}{ll} \Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N \leq^- M \dashv \Theta', \hat\alpha\,[= P] & \text{Subderivation} \\ \Theta, \hat\alpha_2 \vdash [\hat\alpha_2/\alpha]N \leq^- M \dashv \Theta'_2, \hat\alpha_2\,[= P_2] & \text{Subderivation} \\ \Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N \leq^- M \dashv \Theta'_2, \hat\alpha\,[= P_2] & \text{Renaming the free existential variable} \\ \Theta', \hat\alpha[= P] = \Theta'_2, \hat\alpha\,[= P_2] & \text{By i.h.} \\ ☞ \quad \Theta', \hat\alpha[= P] = \Theta'_2, \hat\alpha_2\,[= P_2] & \text{Substituting back the original name } (\hat\alpha = \hat\alpha_2) \end{array}$$

- **Case** $\dfrac{\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha}{\Theta \vdash N \leq^- \forall\alpha.\, M \dashv \Theta'} \leq^\pm\mathsf{Aforallr}$

$$\begin{array}{ll} \Theta \vdash N \leq^- \forall\alpha.\, M \dashv \Theta' & \text{Assumption} \\ \Theta \vdash N \leq^- \forall\alpha.\, M \dashv \Theta'_2 & \text{Assumption} \end{array}$$

By the structure of $\forall\alpha.\, M$, the derivation of the second hypothesis must conclude with an application of $\leq^\pm\mathsf{Aforallr}$.

$$\begin{array}{ll} \Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha & \text{Subderivation} \\ \Theta, \alpha \vdash N \leq^- M \dashv \Theta'_2, \alpha & \text{Subderivation} \\ \Theta', \alpha = \Theta'_2, \alpha & \text{By i.h.} \\ ☞ \quad \Theta' = \Theta'_2 & \text{By above} \end{array}$$

- **Case** $$\dfrac{\Theta \vdash Q \leq^{+} P \dashv \Theta' \qquad \Theta' \vdash [\Theta']N \leq^{-} M \dashv \Theta''}{\Theta \vdash P \rightarrow N \leq^{-} Q \rightarrow M \dashv \Theta''} \leq^{\pm}\mathsf{Aarrow}$$

$\Theta \vdash P \rightarrow N \leq^{-} Q \rightarrow M \dashv \Theta''$	Assumption
$\Theta \vdash P \rightarrow N \leq^{-} Q \rightarrow M \dashv \Theta''_2$	Assumption

By the structure of $P \rightarrow N$ and $Q \rightarrow M$, the derivation of the second hypothesis must conclude with an application of $\leq^{\pm}\mathsf{Aarrow}$.

	$\Theta \vdash Q \leq^{+} P \dashv \Theta'$	Subderivation
	$\Theta \vdash Q \leq^{+} P \dashv \Theta'_2$	Subderivation
	$\Theta' = \Theta'_2$	By i.h.
	$\Theta' \vdash [\Theta']N \leq^{-} M \dashv \Theta''$	Subderivation
	$\Theta'_2 \vdash [\Theta']N \leq^{-} M \dashv \Theta''_2$	Subderivation
	$\Theta' \vdash [\Theta']N \leq^{-} M \dashv \Theta''_2$	Using $\Theta' = \Theta'_2$
☞	$\Theta'' = \Theta''_2$	By i.h.

- **Case** $$\dfrac{\Theta \vdash Q \leq^{+} P \dashv \Theta' \qquad \Theta' \vdash [\Theta']P \leq^{+} Q \dashv \Theta''}{\Theta \vdash \uparrow P \leq^{-} \uparrow Q \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}\uparrow$$

$\Theta \vdash \uparrow P \leq^{-} \uparrow Q \dashv \Theta''$	Assumption
$\Theta \vdash \uparrow P \leq^{-} \uparrow Q \dashv \Theta''_2$	Assumption

By the structure of $\uparrow P$ and $\uparrow Q$, the derivation of the second hypothesis must conclude with an application of $\leq^{\pm}\mathsf{Ashift}\uparrow$.

	$\Theta \vdash Q \leq^{+} P \dashv \Theta'$	Subderivation
	$\Theta \vdash Q \leq^{+} P \dashv \Theta'_2$	Subderivation
	$\Theta' = \Theta'_2$	By i.h.
	$\Theta' \vdash [\Theta']P \leq^{+} Q \dashv \Theta''$	Subderivation
	$\Theta'_2 \vdash [\Theta'_2]P \leq^{+} Q \dashv \Theta''_2$	Subderivation
	$\Theta' \vdash [\Theta']P \leq^{-} Q \dashv \Theta''_2$	Using $\Theta' = \Theta'_2$
☞	$\Theta'' = \Theta''_2$	By i.h.

□

I' Decidability of subtyping

I'.1 Lemmas for decidability

Lemma I.1 (Completed non-ground size bounded by ground size)**.**

- *If* $\Theta \vdash P \leq^{+} Q \dashv \Theta'$, Θ ctx*,* P *ground, and* $[\Theta]Q = Q$*, then* $|[\Theta']Q|_{NQ} \leq |P|_{NQ}$.
- *If* $\Theta \vdash N \leq^{-} M \dashv \Theta'$, Θ ctx*,* M *ground, and* $[\Theta]N = N$*, then* $|[\Theta']N|_{NQ} \leq |M|_{NQ}$.

Proof. Proof sketch by rule induction on algorithmic subtyping judgment. The justification here for using the i.h. omits the reasoning for why the premises of well-formedness must hold for the subderivations if we know that they hold for the conclusion. This reasoning should be identical to that in Lemma E.2 (Algorithmic subtyping is w.f.).

- **Case**

$$\frac{}{\Theta_L, \alpha, \Theta_R \vdash \alpha \leq^+ \alpha \dashv \Theta_L, \alpha, \Theta_R} \leq^\pm\mathsf{Arefl}$$

	$[\Theta_L, \alpha, \Theta_R]\alpha = \alpha$	By definition of $[-]-$
☞	$\lvert[\Theta_L, \alpha, \Theta_R]\alpha\rvert_{NQ} \leq \lvert\alpha\rvert_{NQ}$	Since the types are equal

- **Case**

$$\frac{\Theta_L \vdash P\ \mathsf{type}^+ \qquad P\ \mathsf{ground}}{\Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^+ \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R} \leq^\pm\mathsf{Ainst}$$

	$[\Theta_L, \hat{\alpha} = P, \Theta_R]\hat{\alpha} = P$	By definition of $[-]-$
☞	$\lvert[\Theta_L, \hat{\alpha} = P, \Theta_R]\hat{\alpha}\rvert_{NQ} \leq \lvert P\rvert_{NQ}$	Since the types are equal

- **Case**

$$\frac{\Theta \vdash M \leq^- N \dashv \Theta' \qquad \Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''}{\Theta \vdash {\downarrow}N \leq^+ {\downarrow}M \dashv \Theta''} \leq^\pm\mathsf{Ashift}{\downarrow}$$

	$[\Theta']M$ ground	By well-formedness on the first premise
	Θ' ctx	"
	$\Theta' \longrightarrow \Theta''$	By well-formedness on the first and second premises
	Θ'' ctx	"
	Θ' ctx	Above
	$[\Theta']M$ ground	Above
	$\Theta' \longrightarrow \Theta''$	Above
	Θ'' ctx	Above
	$\lvert[\Theta']M\rvert_{NQ} = \lvert[\Theta'']M\rvert_{NQ}$	By Lemma G.4 (Context extension ground substitution size)
	$\Theta \vdash M \leq^- N \dashv \Theta'$	Subderivation
	$\lvert[\Theta'']M\rvert_{NQ} \leq \lvert[\Theta']M\rvert_{NQ}$	Since the sizes are equal
	$\leq \lvert N\rvert_{NQ}$	By i.h.
☞	$\lvert[\Theta'']{\downarrow}M\rvert_{NQ} \leq \lvert{\downarrow}N\rvert_{NQ}$	By definition of $\lvert-\rvert_{NQ}$

- **Case**

$$\frac{\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha}{\Theta \vdash N \leq^- \forall\alpha. M \dashv \Theta'} \leq^\pm\mathsf{Aforallr}$$

$\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha$	Subderivation
$\lvert[\Theta', \alpha]N\rvert_{NQ} \leq \lvert M\rvert_{NQ}$	By i.h.
$[\Theta', \alpha]N = [\Theta']N$	By definition of $[-]-$

$$|M|_{NQ} = |\forall\alpha.\, M|_{NQ} \qquad \text{By definition of } |-|_{NQ}$$

☞ $$|[\Theta']N|_{NQ} \le |\forall\alpha.\, M|_{NQ} \qquad \text{Substituting above}$$

- **Case** $$\dfrac{\Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N \le^- M \dashv \Theta', \hat\alpha\,[= P] \qquad M \ne \forall\alpha.\, M'}{\Theta \vdash \forall\alpha.\, N \le^- M \dashv \Theta'} \le^{\pm}\mathsf{AforallI}$$

$$\Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N \le^- M \dashv \Theta', \hat\alpha\,[= P] \qquad \text{Subderivation}$$

$$|[\Theta', \hat\alpha\,[= P]][\hat\alpha/\alpha]N|_{NQ} \le |M|_{NQ} \qquad \text{By i.h.}$$

Case $\alpha \notin FUV(N)$:

$$[\Theta', \hat\alpha\,[= P]][\hat\alpha/\alpha]N = [\Theta']N \qquad \text{By definition of } [-]-$$

$$|[\Theta']N|_{NQ} \le |M|_{NQ} \qquad \text{Substituting above}$$

☞ $$|[\Theta']\forall\alpha.\, N|_{NQ} \le |M|_{NQ} \qquad \text{By definition of } |-|_{NQ}$$

Case $\alpha \in FUV(N)$:

$\Theta', \hat\alpha\,[= P]$ ctx — By Lemma E.2 (Algorithmic subtyping is w.f.)

$[\Theta', \hat\alpha\,[= P]][\hat\alpha/\alpha]N$ ground — ″

$(\Theta', \hat\alpha\,[= P]) = (\Theta', \hat\alpha = P)$ — $\alpha \in FUV(N)$ so $\hat\alpha \in FEV([\hat\alpha/\alpha]N)$. Since $\hat\alpha$ is not ground, the context must solve $\hat\alpha$ to make $[\Theta', \hat\alpha\,[= P]][\hat\alpha/\alpha]N$ ground.

P ground — Inversion (Cwfsolvedguess)

$$
\begin{aligned}
[\Theta', \hat\alpha = P][\hat\alpha/\alpha]N &= [\Theta'][P/\hat\alpha][\hat\alpha/\alpha]N && \text{By definition of } [-]- \\
&= [\Theta'][P/\alpha]N && \text{By definition of } [-]- \\
&= [[\Theta']P/\alpha][\Theta']N && \text{Since the type being replaced is a universal variable} \\
&= [P/\alpha][\Theta']N && \text{Since } P \text{ is ground} \\
|[\Theta']\forall\alpha.\, N|_{NQ} &= |[\Theta']N|_{NQ} && \text{By definition of } |-|_{NQ} \\
&\le |[P/\alpha][\Theta']N|_{NQ} && \text{The additional substitution cannot decrease the size of the type} \\
&= |[\Theta', \hat\alpha = P][\hat\alpha/\alpha]N|_{NQ} && \text{Above} \\
&\le |M|_{NQ} && \text{Above}
\end{aligned}
$$

☞ $$|[\Theta']\forall\alpha.\, N|_{NQ} \le |M|_{NQ} \qquad \text{By transitivity of } \le$$

- **Case** $$\dfrac{\Theta \vdash Q \le^+ P \dashv \Theta' \qquad \Theta' \vdash [\Theta']N \le^- M \dashv \Theta''}{\Theta \vdash P \to N \le^- Q \to M \dashv \Theta''} \le^{\pm}\mathsf{Aarrow}$$

$\Theta \vdash Q \le^+ P \dashv \Theta'$ — Subderivation

$|[\Theta']P|_{NQ} \le |Q|_{NQ}$ — By i.h.

$[\Theta']P$ ground — By w.f. applied to first subderivation

$\Theta' \longrightarrow \Theta''$ — By w.f. applied to second subderivation

$|[\Theta']P|_{NQ} = |[\Theta'']P|_{NQ}$ — By Lemma G.4 (Context extension ground substitution size)

$\Theta \vdash [\Theta']N \le^- M \dashv \Theta''$ — Subderivation

$|[\Theta''][\Theta']N|_{NQ} \le |M|_{NQ}$ — By i.h.

$$|[\Theta''][\Theta']N|_{NQ} = |[\Theta'']N|_{NQ}$$ By Lemma G.3 (Context extension substitution size)

$$\begin{aligned}|[\Theta''](P \to N)|_{NQ} &= |[\Theta'']P|_{NQ} + |[\Theta'']N|_{NQ} + 1 && \text{By definition of } |-|_{NQ} \\ &= |[\Theta']P|_{NQ} + |[\Theta''][\Theta']N|_{NQ} + 1 && \text{Substituting above} \\ &\leq |Q|_{NQ} + |M|_{NQ} + 1 && \text{Using above inequalities}\end{aligned}$$

☞ $|[\Theta''](P \to N)|_{NQ} \leq |Q \to M|_{NQ}$ By definition of $|-|_{NQ}$

- **Case** $\dfrac{\Theta \vdash Q \leq^{+} P \dashv \Theta' \qquad \Theta' \vdash [\Theta']P \leq^{+} Q \dashv \Theta''}{\Theta \vdash \uparrow P \leq^{-} \uparrow Q \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}\uparrow$

  Symmetric to $\leq^{\pm}\mathsf{Ashift}\downarrow$ case.

□

I'.2 Statement

Lemma I.2 (Decidability of algorithmic subtyping)**.** *There exists a total order $\sqsubset$ on well-formed algorithmic subtyping judgments such that for each derivation with subtyping judgment premises A_i and conclusion B, each A_i compares less than B, i.e.* $\forall i.\, A_i \sqsubset B$.

Proof. The ordering is the same lexicographic ordering we used earlier in Lemma B.7 (Declarative subtyping is transitive) and Theorem G.5 (Completeness of algorithmic subtyping):

- $(|P|_{NQ}, \mathrm{NPQ}(P) + \mathrm{NPQ}(Q))$ for positive judgments $\Theta \vdash P \leq^{+} Q \dashv \Theta'$
- $(|M|_{NQ}, \mathrm{NPQ}(M) + \mathrm{NPQ}(N))$ for negative judgments $\Theta \vdash N \leq^{-} M \dashv \Theta'$

In this ordering, $\mathrm{NPQ}(A)$ is the number of prenex universal quantifiers in the type A and $|A|_{NQ}$ is the size of the algorithmic type A defined in Lemma I.1 (Completed non-ground size bounded by ground size) (N.B. universal quantifiers do not contribute to this size).

Sketch of this proof: We will prove by rule induction that each subderivation compares less than each conclusion for each derivation of the algorithmic subtyping judgment. We will assume the following additional statements about the judgment being proved in the rule induction (the same assumptions used in Lemma E.2 (Algorithmic subtyping is w.f.)):

- For positive subderivations $\Theta \vdash P \leq^{+} Q \dashv \Theta'$:
  1. Θ ctx
  2. P ground
  3. $[\Theta]Q = Q$
- For negative subderivations $\Theta \vdash N \leq^{-} M \dashv \Theta'$:
  1. Θ ctx
  2. M ground
  3. $[\Theta]N = N$

The subtyping algorithm should first check that these well-formedness assumptions hold for the judgment in question. By the same argument as used in Lemma E.2 (Algorithmic subtyping is w.f.), we can show that they are preserved by the algorithmic subtyping rules from conclusion to subderivations.

- Checking the well-formedness of a type is decidable:
  - Typing contexts are finite, so checking UV and EV is decidable.
  - There is exactly one type well-formedness rule to apply for each type.
  - The application of each rule reduces the natural size of the type (same as $|\_|_{NQ}$ except universal quantification contributes to this size).

- Checking whether a type is ground is decidable, since types are finite.
- Checking context well-formedness is decidable.
  - Checking type well-formedness is decidable.
  - Checking whether a type is ground is decidable.
  - There is exactly one rule to apply for each context.
  - The application of each rule for each non-empty context reduces the number of items in the context by 1.
- Applying a context as a substitution to a type is decidable since each rule decreases the lexicographic order (number of free existential variables, number of items in the context). This follows from the requirement that solutions to existential variables are ground.

The subtyping algorithm should then proceed to try and apply algorithmic subtyping rules based on the structure of the types until there are no more subderivations to prove. The structure of the types dictate a single rule to apply at each stage. We now sketch a proof that each derivation of an algorithmic subtyping judgment is finite. As with Lemma I.1 (Completed non-ground size bounded by ground size), we skip justifications for why the same assumptions used in Lemma E.2 (Algorithmic subtyping is w.f.) continue to hold.

The key idea is that at each step in the proof, the subderivation either fails or the algorithm determines that it is derivable. If the first subderivation fails, the algorithm should terminate in a failure state, and therefore we do not need to prove anything about the second subderivation. This allows us to use the first subderivations of the shift rules in the proof that the second subderivations are smaller than the conclusions. We have omitted stating this reasoning in each of the proof cases.

- **Case** $$\frac{}{\Theta_L, \alpha, \Theta_R \vdash \alpha \leq^+ \alpha \dashv \Theta_L, \alpha, \Theta_R} \leq^{\pm}\mathsf{Arefl}$$

  No algorithmic subtyping subderivations.

- **Case** $$\frac{\Theta_L \vdash P \ \mathsf{type}^+ \qquad P \ \text{ground}}{\Theta_L, \hat{\alpha}, \Theta_R \vdash P \leq^+ \hat{\alpha} \dashv \Theta_L, \hat{\alpha} = P, \Theta_R} \leq^{\pm}\mathsf{Ainst}$$

  No algorithmic subtyping subderivations.

- **Case** $$\frac{\Theta \vdash M \leq^- N \dashv \Theta' \qquad \Theta' \vdash N \leq^- [\Theta']M \dashv \Theta''}{\Theta \vdash \downarrow N \leq^+ \downarrow M \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}\downarrow$$

  $$\begin{aligned} |N|_{NQ} &= |\downarrow N|_{NQ} - 1 && \text{By definition of } |-|_{NQ} \\ &< |\downarrow N|_{NQ} \end{aligned}$$

Therefore the first subderivation compares less than the conclusion.

$$\begin{aligned} \Theta \vdash M \leq^- N \dashv \Theta' & && \text{Subderivation} \\ |[\Theta']M|_{NQ} &\leq |N|_{NQ} && \text{By Lemma I.1 (Completed non-ground size bounded by ground size)} \\ &< |\downarrow N|_{NQ} && \text{By definition of } |-|_{NQ} \end{aligned}$$

Therefore the second subderivation compares less than the conclusion.

- **Case** $\dfrac{\Theta, \alpha \vdash N \leq^- M \dashv \Theta', \alpha}{\Theta \vdash N \leq^- \forall\alpha.\, M \dashv \Theta'} \leq^{\pm}\mathsf{Aforallr}$

$$\begin{aligned} |M|_{NQ} &= |\forall\alpha.\, M|_{NQ} && \text{By definition of } |{-}|_{NQ} \\ NPQ(M) &< NPQ(\forall\alpha.\, M) && \text{The LHS has one fewer prenex quantifier} \\ NPQ(N) + NPQ(M) &< NPQ(N) + NPQ(\forall\alpha.\, M) \end{aligned}$$

Therefore the subderivation compares less than the conclusion.

- **Case** $\dfrac{\Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N \leq^- M \dashv \Theta', \hat\alpha\,[= P] \qquad M \neq \forall\alpha.\, M'}{\Theta \vdash \forall\alpha.\, N \leq^- M \dashv \Theta'} \leq^{\pm}\mathsf{Aforalll}$

$$\begin{aligned} |M|_{NQ} &= |M|_{NQ} \\ NPQ([\hat\alpha/\alpha]N) &< NPQ(\forall\alpha.\, N) && \text{The LHS has one fewer prenex quantifier} \end{aligned}$$

Therefore the subderivation compares less than the conclusion.

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \dashv \Theta' \qquad \Theta' \vdash [\Theta']N \leq^- M \dashv \Theta''}{\Theta \vdash P \to N \leq^- Q \to M \dashv \Theta''} \leq^{\pm}\mathsf{Aarrow}$

$$\begin{aligned} |Q|_{NQ} &= |Q \to M|_{NQ} - |M|_{NQ} - 1 && \text{By definition of } |{-}|_{NQ} \\ &< |Q \to M|_{NQ} \end{aligned}$$

Therefore the first subderivation compares less than the conclusion.

$$\begin{aligned} |M|_{NQ} &= |Q \to M|_{NQ} - |Q|_{NQ} - 1 && \text{By definition of } |{-}|_{NQ} \\ &< |Q \to M|_{NQ} \end{aligned}$$

Therefore the second subderivation compares less than the conclusion.

- **Case** $\dfrac{\Theta \vdash Q \leq^+ P \dashv \Theta' \qquad \Theta' \vdash [\Theta']P \leq^+ Q \dashv \Theta''}{\Theta \vdash \uparrow P \leq^- \uparrow Q \dashv \Theta''} \leq^{\pm}\mathsf{Ashift}\uparrow$

$$\begin{aligned} |Q|_{NQ} &= |\uparrow Q|_{NQ} - 1 && \text{By definition of } |{-}|_{NQ} \\ &< |\uparrow Q|_{NQ} \end{aligned}$$

Therefore the first subderivation compares less than the conclusion.

$$\begin{aligned} &\Theta' \vdash Q \leq^+ P \dashv \Theta' && \text{Subderivation} \\ |[\Theta']P|_{NQ} &\leq |Q|_{NQ} && \text{By Lemma I.1 (Completed non-ground size bounded by ground size)} \\ &< |\uparrow Q|_{NQ} && \text{By definition of } |{-}|_{NQ} \end{aligned}$$

Therefore the second subderivation compares less than the conclusion.

□

J' Isomorphic types

Lemma J.1 (Isomorphic environments type the same terms). *If* $\Theta \vdash \Gamma \cong \Gamma'$*, then:*

- *If* $\Theta; \Gamma \vdash v : P$ *then* $\exists P'$ *such that* $\Theta \vdash P \cong^- P'$ *and* $\Theta; \Gamma' \vdash v : P'$.
- *If* $\Theta; \Gamma \vdash t : N$ *then* $\exists N'$ *such that* $\Theta \vdash N \cong^- N'$ *and* $\Theta; \Gamma' \vdash t : N'$.
- *If* $\Theta; \Gamma \vdash s : N \gg M$ *and* $\Theta \vdash N \cong^- N'$*, then* $\exists M'$ *such that* $\Theta \vdash M \cong^- M'$ *and* $\Theta; \Gamma \vdash s : N' \gg M'$.

Proof. By mutual induction on the checking, synthesis, and spine judgments.
We first define notions of the sizes of terms and spines:

$\boxed{|e|}$ The size of the term e

$\boxed{|s|}$ The size of the spine s

$$|x| = 1 \qquad |\{t\}| = |t| + 1$$
$$|\lambda x.t| = |t| + 1 \qquad |\Lambda\alpha.\,t| = |t| + 1$$
$$|\mathsf{return}\ v| = |v| + 1$$
$$|\mathsf{let}\ x : P = f(s); t| = |f| + |s| + |t| + 1 \qquad |\mathsf{let}\ x = f(s); t| = |f| + |s| + |t| + 1$$
$$|\epsilon| = 1 \qquad |s, v| = |s| + |v| + 1$$

We perform the induction using the following metric on judgments:

$\boxed{|J|}$ The size of the judgment J

$$|\Theta; \Gamma \vdash v : P \dashv \Theta'| = (|f|, 0)$$
$$|\Theta; \Gamma \vdash t : N \dashv \Theta'| = (|t|, 0)$$
$$|\Theta; \Gamma \vdash t : N \gg M \dashv \Theta'| = (|t|, \mathrm{NPQ}(N))$$

- **Case** $\dfrac{x : P \in \Gamma}{\Theta; \Gamma \vdash x : P}$ Dvar

	$x : P \in \Gamma$	Premise
	$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
(1)	$x : P' \in \Gamma'$	Inversion (Eisovar)
☞	$\Theta \vdash P \cong^+ P'$	"
☞	$\Theta; \Gamma' \vdash x : P'$	By Dvar and (1)

- **Case** $\dfrac{\Theta; \Gamma, x : P \vdash t : N}{\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N}$ Dλabs

$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
$\Theta \vdash P \cong^{+} P$	By Lemma B.1 (Declarative subtyping is reflexive)
$\Theta \vdash \Gamma, x : P \cong \Gamma', x : P$	By Eisovar
$\Theta; \Gamma, x : P \vdash t : N$	Subderivation
$\Theta \vdash N \cong^{-} N'$	By i.h. (term size has decreased)
$\Theta; \Gamma', x : P \vdash t : N'$	"
☞ $\Theta \vdash P \to N \cong^{-} P \to N'$	By $\leq^{\pm}$Darrow
☞ $\Theta; \Gamma' \vdash \lambda x : P.\, t : P \to N'$	By Dλabs

- **Case** $\dfrac{\Theta, \alpha; \Gamma \vdash t : N}{\Theta; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N}$ Dgen

$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
$\Theta, \alpha; \Gamma \vdash t : N$	Subderivation
$\Theta \vdash N \cong^{-} N'$	By i.h. (term size has decreased)
$\Theta, \alpha; \Gamma' \vdash t : N'$	"
☞ $\Theta \vdash \forall\alpha.\, N \cong^{+} \forall\alpha.\, N'$	By $\leq^{\pm}$Dforalll (using $P = \alpha$) and $\leq^{\pm}$Dforallr
☞ $\Theta; \Gamma' \vdash \Lambda\alpha.\, t : \forall\alpha.\, N'$	By Dgen

- **Case** $\dfrac{\Theta; \Gamma \vdash t : N}{\Theta; \Gamma \vdash \{t\} : {\downarrow}N}$ Dthunk

$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
$\Theta; \Gamma \vdash t : N$	Subderivation
$\Theta \vdash N \cong^{-} N'$	By i.h. (term size has decreased)
$\Theta; \Gamma' \vdash t : N'$	"
☞ $\Theta \vdash {\downarrow}N \cong^{-} {\downarrow}N'$	By $\leq^{\pm}$Dshift↓
☞ $\Theta; \Gamma' \vdash \{t\} : {\downarrow}N'$	By Dthunk

- **Case** $\dfrac{\Theta; \Gamma \vdash v : P}{\Theta; \Gamma \vdash \mathsf{return}\ v : {\uparrow}P}$ Dreturn

$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
$\Theta; \Gamma \vdash v : P$	Subderivation
$\Theta \vdash P \cong^{+} P'$	By i.h. (term size has decreased)
$\Theta; \Gamma' \vdash v : P'$	"

☞ $\Theta \vdash \uparrow P \cong^- \uparrow P'$ By $\leq^\pm$Dshift$\uparrow$

☞ $\Theta; \Gamma' \vdash \text{return } v : \uparrow P'$ By Dreturn

- **Case** $$\dfrac{\Theta;\Gamma \vdash v : \downarrow M \qquad \Theta;\Gamma \vdash s : M \gg \uparrow Q \qquad \Theta \vdash \uparrow P \leq^- \uparrow Q \qquad \Theta;\Gamma, x : P \vdash t : N}{\Theta;\Gamma \vdash \text{let } x : P = v(s); t : N}\text{Dambiguouslet}$$

	$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
	$\Theta;\Gamma \vdash v : \downarrow M'$	Subderivation
	$\Theta \vdash \downarrow M \cong^+ \downarrow M'$	By i.h. (term size has decreased)
(1)	$\Theta;\Gamma' \vdash v : \downarrow M'$	"
	$\Theta;\Gamma \vdash s : M \gg \uparrow Q$	Subderivation
	$\Theta \vdash M \cong^- M'$	Inversion ($\leq^\pm$Dshift$\downarrow$)
	$\Theta \vdash \uparrow Q \cong^- \uparrow Q'$	By i.h. (term size has decreased)
(2)	$\Theta;\Gamma' \vdash s : M' \gg \uparrow Q'$	"
	$\Theta \vdash \uparrow P \leq^- \uparrow Q$	Premise
(3)	$\Theta \vdash \uparrow P \leq^- \uparrow Q'$	By Lemma B.7 (Declarative subtyping is transitive)
	$\Theta \vdash \Gamma, x : P \cong \Gamma', x : P$	By Eisovar
	$\Theta;\Gamma, x : P \vdash t : N$	Subderivation
☞	$\Theta \vdash N \cong^- N'$	By i.h. (term size has decreased)
(4)	$\Theta;\Gamma', x : P \vdash t : N'$	"
☞	$\Theta;\Gamma' \vdash \text{let } x : P = v(s); t : N'$	By Dambiguouslet and (1–4)

- **Case** $$\dfrac{\Theta;\Gamma \vdash s : M \gg \uparrow Q \qquad \Theta;\Gamma, x : Q \vdash t : N \qquad \begin{array}{c}\Theta;\Gamma \vdash v : \downarrow M \\ \forall P.\ \text{if } \Theta;\Gamma \vdash s : M \gg \uparrow P \text{ then } \Theta \vdash Q \cong^+ P\end{array}}{\Theta;\Gamma \vdash \text{let } x = v(s); t : N}\text{Dunambiguouslet}$$

	$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
	$\Theta;\Gamma \vdash v : \downarrow M$	Subderivation
	$\Theta \vdash \downarrow M \cong^+ \downarrow M'$	By i.h. (term size has decreased)
(1)	$\Theta;\Gamma' \vdash v : \downarrow M'$	"
	$\Theta;\Gamma \vdash s : M \gg \uparrow Q$	Subderivation
	$\Theta \vdash M \cong^- M'$	Inversion ($\leq^\pm$Dshift$\downarrow$)
	$\Theta \vdash \uparrow Q \cong^- \uparrow Q'$	By i.h. (term size has decreased)
(2)	$\Theta;\Gamma' \vdash s : M' \gg \uparrow Q'$	"
	$\Theta \vdash Q \cong^+ Q'$	Inversion ($\leq^\pm$Dshift$\uparrow$)
	$\Theta \vdash \Gamma, x : Q \cong \Gamma', x : Q'$	By Eisovar
☞	$\Theta \vdash N \cong^- N'$	By i.h. (term size has decreased)
(3)	$\Theta;\Gamma', x : Q' \vdash t : N'$	"

To show the final premise of the Dunambiguouslet rule, let P be arbitrary and assume $\Theta; \Gamma' \vdash s : M' \gg \uparrow P$. Now show that $\Theta \vdash Q' \cong^- P$:

$\Theta; \Gamma' \vdash s : M' \gg \uparrow P$	Assumption
$\Theta \vdash \Gamma \cong \Gamma'$	Above
$\Theta \vdash M \cong^- M'$	Above
$\Theta; \Gamma \vdash s : M \gg \uparrow P'$	By i.h. (term size has decreased)
$\Theta \vdash \uparrow P \cong^- \uparrow P'$	"
$\Theta \vdash P \cong^- P'$	Inversion ($\leq^{\pm}$Dshift$\uparrow$)
$\Theta \vdash Q \cong^- P'$	Applying subderivation
$\Theta \vdash Q \cong^- Q'$	Above
$\Theta \vdash Q' \cong^- P$	By Lemma B.7 (Declarative subtyping is transitive)

We have now shown the final premise of Dunambiguouslet, so apply it to give the required typing judgment:

☞ $\Theta; \Gamma' \vdash \text{let } x : P = v(s); t : N'$ — By Dunambiguouslet and (1–3)

- **Case**

$$\frac{}{\Theta; \Gamma \vdash \epsilon : N \gg N}\ \text{Dspinenil}$$

	$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
☞	$\Theta \vdash N \cong^- N'$	Assumption
☞	$\Theta; \Gamma' \vdash \epsilon : N' \gg N'$	By Dspinenil

- **Case**

$$\frac{\Theta; \Gamma \vdash v : P \qquad \Theta \vdash P \leq^+ Q \qquad \Theta; \Gamma \vdash s : N \gg M}{\Theta; \Gamma \vdash v, s : (Q \to N) \gg M}\ \text{Dspinecons}$$

$\Theta \vdash \Gamma \cong \Gamma'$	Assumption
$\Theta \vdash Q \to N \cong^- T$	Assumption

By inversion, $T = \forall\alpha \cdots \forall\beta. P' \to N'$. Therefore perform induction over the number of prenex universal quantifiers, n:

- **Case** $n = 0$:

	$N = Q' \to N'$	By inversion
	$\Theta \vdash Q \to N \cong^- Q' \to N'$	Assumption
	$\Theta \vdash Q \cong^+ Q'$	By inversion
	$\Theta \vdash N \cong^- N'$	By inversion
	$\Theta \vdash P \cong^+ P'$	By outer i.h. (term size has decreased)
(1)	$\Theta; \Gamma' \vdash v : P'$	"
	$\Theta \vdash P \leq^+ Q$	Subderivation

(2) $\Theta \vdash P' \leq^{+} Q'$ By Lemma B.7 (Declarative subtyping is transitive)

$\Theta; \Gamma \vdash s : N \gg M$ Subderivation

☞ $\Theta \vdash M \cong^{-} M'$ By outer i.h. (term size has decreased)

(3) $\Theta; \Gamma' \vdash s : N' \gg M'$ ″

☞ $\Theta; \Gamma' \vdash v, s : Q' \to N' \gg M'$ By Dspinecons and (1–3)

– **Case:** $n = k + 1$

$T = \forall \alpha. T'$ By inversion

$\Theta \vdash P \to N \cong^{+} [P/\alpha]T'$ By inversion ($\leq^{\pm}$Dforalll, $\leq^{\pm}$Dforallr), for $\Theta \vdash P\ \mathsf{type}^{+}$

$\Theta; \Gamma' \vdash v, s : [P/\alpha]T' \gg M'$ By inner i.h.

☞ $\Theta \vdash M \cong^{-} M'$ ″

$\Theta; \Gamma' \vdash v, s : (\forall \alpha. T') \gg M'$ By Dspinetypeabs

☞ $\Theta; \Gamma' \vdash v, s : T \gg M'$ By equality

- **Case** $$\dfrac{\Theta \vdash P\ \mathsf{type}^{+} \qquad \Theta; \Gamma \vdash s : [P/\alpha]N \gg M}{\Theta; \Gamma \vdash s : (\forall \alpha. N) \gg M}\ \mathsf{Dspinetypeabs}$$

$\Theta \vdash \Gamma \cong \Gamma'$ Assumption

$\Theta \vdash \forall \alpha. N \cong^{-} N''$ Assumption

$N'' = \forall \beta. N'$ Inversion ($\leq^{\pm}$Dforalll/ $\leq^{\pm}$Dforallr)

$\Theta \vdash [P/\alpha]N \cong^{-} [R/\beta]N'$ ″, for $\Theta \vdash R\ \mathsf{type}^{+}$

☞ $\Theta \vdash M \cong^{-} M'$ By i.h. (term size is the same and the number of prenex quantifiers has decreased)

$\Theta; \Gamma' \vdash s : [R/\beta]N' \gg M'$ ″

$\Theta; \Gamma' \vdash s : \forall \beta. N' \gg M'$ By Dspinetypeabs

☞ $\Theta; \Gamma' \vdash s : N'' \gg M'$ By definition of N''

□

K' Well-formedness of typing

Lemma K.1 (Well-formedness of restricted contexts). *If* Θ ctx, Θ' ctx, $\Theta \implies \Theta'$, *then* $\Theta' \upharpoonright \Theta$ ctx, $\Theta \longrightarrow \Theta' \upharpoonright \Theta$, *and* $\Theta' \upharpoonright \Theta \implies \Theta'$.

Proof. By rule induction on the $\Theta \implies \Theta'$ judgment.

- **Case** $$\dfrac{}{\cdot \implies \cdot}\ \mathsf{Wcempty}$$

	$\cdot \upharpoonright \cdot = \cdot$	By $\upharpoonright$empty
☞	$\cdot$ ctx	By Cwfempty
☞	$\cdot \longrightarrow \cdot$	By Cempty
☞	$\cdot \Longrightarrow \cdot$	By Wcempty

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \alpha \Longrightarrow \Theta', \alpha}$ Wcuvar

	Θ, α ctx	Assumption
	Θ ctx	Inversion (Cwfuvar)
	Θ', α ctx	Assumption
	Θ' ctx	Inversion (Cwfuvar)
	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta \longrightarrow \Theta' \upharpoonright \Theta$	By i.h.
	$\Theta' \upharpoonright \Theta$ ctx	"
	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta'$	"
	$(\Theta', \alpha) \upharpoonright (\Theta, \alpha) = (\Theta' \upharpoonright (\Theta, \alpha)), \alpha$	By $\upharpoonright$uvar
☞	$(\Theta' \upharpoonright \Theta), \alpha$ ctx	By Cwfuvar
☞	$\Theta, \alpha \longrightarrow (\Theta' \upharpoonright \Theta), \alpha$	By Cuvar
☞	$(\Theta' \upharpoonright \Theta), \alpha \Longrightarrow \Theta', \alpha$	By Wcuvar

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha}}$ Wcunsolvedguess

	$\Theta, \hat{\alpha}$ ctx	Assumption
	Θ ctx	Inversion (Cwfunsolvedguess)
	$\Theta', \hat{\alpha}$ ctx	Assumption
	Θ' ctx	Inversion (Cwfunsolvedguess)
	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta' \upharpoonright \Theta$ ctx	By i.h.
	$\Theta \longrightarrow \Theta' \upharpoonright \Theta$	"
	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta'$	"
	$(\Theta', \hat{\alpha}) \upharpoonright (\Theta, \hat{\alpha}) = (\Theta' \upharpoonright \Theta), \hat{\alpha}$	By $\upharpoonright$guessin
☞	$(\Theta' \upharpoonright \Theta), \hat{\alpha}$ ctx	By Cwfunsolvedguess
☞	$\Theta, \hat{\alpha} \longrightarrow (\Theta' \upharpoonright \Theta), \hat{\alpha}$	By Cunsolvedguess
☞	$(\Theta' \upharpoonright \Theta), \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha}$	By Wcunsolvedguess

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha} = P}$ Wcsolveguess

	$\Theta, \hat{\alpha}\ \mathsf{ctx}$	Assumption
	$\Theta\ \mathsf{ctx}$	Inversion (Cwfunsolvedguess)
	$\Theta', \hat{\alpha} = P\ \mathsf{ctx}$	Assumption
	$\Theta'\ \mathsf{ctx}$	Inversion (Cwfsolvedguess)
	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta' \upharpoonright \Theta\ \mathsf{ctx}$	By i.h.
	$\Theta \longrightarrow \Theta' \upharpoonright \Theta$	"
	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta'$	"
	$(\Theta', \hat{\alpha} = P) \upharpoonright (\Theta, \hat{\alpha}) = (\Theta' \upharpoonright \Theta), \hat{\alpha} = P$	By $\upharpoonright$guessin
☞	$(\Theta' \upharpoonright \Theta), \hat{\alpha} = P\ \mathsf{ctx}$	By Cwfsolvedguess
☞	$\Theta, \hat{\alpha} \longrightarrow (\Theta' \upharpoonright \Theta), \hat{\alpha} = P$	By Csolveguess
☞	$(\Theta' \upharpoonright \Theta), \hat{\alpha} = P \Longrightarrow \Theta', \hat{\alpha} = P$	By Wcsolvedguess

- **Case** $\dfrac{\Theta \Longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \Longrightarrow \Theta', \hat{\alpha} = Q}$ Wcsolvedguess

	$\Theta, \hat{\alpha} = P\ \mathsf{ctx}$	Assumption
	$\Theta\ \mathsf{ctx}$	Inversion (Cwfsolvedguess)
	$\Theta', \hat{\alpha} = Q\ \mathsf{ctx}$	Assumption
	$\Theta'\ \mathsf{ctx}$	Inversion (Cwfsolvedguess)
	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta' \upharpoonright \Theta\ \mathsf{ctx}$	By i.h.
	$\Theta \longrightarrow \Theta' \upharpoonright \Theta$	"
	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta'$	"
	$(\Theta', \hat{\alpha} = Q) \upharpoonright (\Theta, \hat{\alpha} = P) = (\Theta' \upharpoonright \Theta), \hat{\alpha} = Q$	By $\upharpoonright$guessin
☞	$(\Theta' \upharpoonright \Theta), \hat{\alpha} = Q\ \mathsf{ctx}$	By Cwfsolvedguess
	$\|\Theta\| \vdash P \cong^{+} Q$	Premise
☞	$\Theta, \hat{\alpha} = P \longrightarrow (\Theta' \upharpoonright \Theta), \hat{\alpha} = Q$	By Csolvedguess
☞	$(\Theta' \upharpoonright \Theta), \hat{\alpha} = Q \Longrightarrow \Theta', \hat{\alpha} = Q$	By Wcsolvedguess

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha}}$ Wcnewunsolvedguess

$\hat{\alpha}\,[= P] \notin \Theta$	Since $\hat{\alpha}$ fresh
$(\Theta', \hat{\alpha}) \upharpoonright \Theta = \Theta' \upharpoonright \Theta$	By $\upharpoonright$guessnotin

	Θ ctx	Assumption
	$\Theta', \hat{\alpha}$ ctx	Assumption
	Θ' ctx	Inversion (Cwfunsolvedguess)
	$\Theta \Longrightarrow \Theta'$	Subderivation
☞	$\Theta' \upharpoonright \Theta$ ctx	By i.h.
☞	$\Theta \longrightarrow \Theta' \upharpoonright \Theta$	"
	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta'$	"
☞	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta', \hat{\alpha}$	By Wcnewunsolvedguess

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha} = P}$ Wcnewsolvedguess

	$\hat{\alpha}\,[= Q] \notin \Theta$	Since $\hat{\alpha}$ fresh
	$(\Theta', \hat{\alpha} = P) \upharpoonright \Theta = \Theta' \upharpoonright \Theta$	By $\upharpoonright$guessnotin
	Θ ctx	Assumption
	$\Theta', \hat{\alpha} = P$ ctx	Assumption
	Θ' ctx	Inversion (Cwfsolvedguess)
	$\Theta \Longrightarrow \Theta'$	Subderivation
☞	$\Theta' \upharpoonright \Theta$ ctx	By i.h.
☞	$\Theta \longrightarrow \Theta' \upharpoonright \Theta$	"
	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta'$	"
☞	$\Theta' \upharpoonright \Theta \Longrightarrow \Theta', \hat{\alpha} = P$	By Wcnewsolvedguess

□

Lemma K.2 (Type well-formed with type variable removed). *If* $\Theta_L, \alpha, \Theta_R \vdash T\,\text{type}^{\pm}$ *and* $\alpha \notin \text{FUV}(T)$*, then* $\Theta_L, \Theta_R \vdash T\,\text{type}^{\pm}$.

Proof. By rule induction over the definition of well-formed types.

- **Case** $\dfrac{\beta \in \text{FUV}(\Theta_L, \alpha, \Theta_R)}{\Theta_L, \alpha, \Theta_R \vdash \beta\,\text{type}^{+}}$ Twfuvar

	$\alpha \notin \text{FUV}(\beta)$	Assumption
	$\beta \neq \alpha$	By above
	$\Theta_L, \alpha, \Theta_R \vdash \beta\ \text{type}^{+}$	Assumption
	$\beta \in \text{FUV}(\Theta_L, \Theta_R)$	By above two statements
☞	$\Theta_L, \Theta_R \vdash \beta\ \text{type}^{+}$	By Twfuvar

- **Case** $\dfrac{\hat{\alpha} \in \mathrm{FEV}(\Theta_L, \alpha, \Theta_R)}{\Theta_L, \alpha, \Theta_R \vdash \hat{\alpha}\ \mathsf{type}^+}$ Twfguess

$\Theta_L, \alpha, \Theta_R \vdash \hat{\alpha}\ \mathsf{type}^+$	Assumption
$\hat{\alpha} \in \mathrm{FEV}(\Theta_L, \Theta_R)$	By above
☞ $\Theta_L, \Theta_R \vdash \hat{\alpha}\ \mathsf{type}^+$	By Twfguess

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_R \vdash {\downarrow}N\ \mathsf{type}^+}$ Twfshift↓

$\Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-$	Premise
$\Theta_L, \Theta_R \vdash N\ \mathsf{type}^-$	By i.h.
☞ $\Theta_L, \Theta_R \vdash {\downarrow}N\ \mathsf{type}^+$	By Twfshift↓

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R, \beta \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_R \vdash \forall\beta.N\ \mathsf{type}^-}$ Twfforall

$\beta \neq \alpha$	β is fresh
$\Theta_L, \alpha, \Theta_R, \beta \vdash N\ \mathsf{type}^-$	Premise
$\Theta_L, \Theta_R, \beta \vdash N\ \mathsf{type}^-$	By i.h.
☞ $\Theta_L, \Theta_R \vdash \forall\beta.N\ \mathsf{type}^-$	By Twfforall

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash P\ \mathsf{type}^+ \qquad \Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_R \vdash P \rightarrow N\ \mathsf{type}^-}$ Twfarrow

$\Theta_L, \alpha, \Theta_R \vdash P\ \mathsf{type}^+$	Premise
$\Theta_L, \Theta_R \vdash P\ \mathsf{type}^+$	By i.h.
$\Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-$	Premise
$\Theta_L, \Theta_R \vdash N\ \mathsf{type}^-$	By i.h.
☞ $\Theta_L, \Theta_R \vdash P \rightarrow N\ \mathsf{type}^-$	By Twfarrow

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash P\ \mathsf{type}^+}{\Theta_L, \alpha, \Theta_R \vdash {\uparrow}P\ \mathsf{type}^-}$ Twfshift↑

$\Theta_L, \alpha, \Theta_R \vdash P\ \mathsf{type}^+$	Premise
$\Theta_L, \Theta_R \vdash P\ \mathsf{type}^+$	By i.h.
☞ $\Theta_L, \Theta_R \vdash {\uparrow}P\ \mathsf{type}^-$	By Twfshift↑

□

Lemma K.3 (Substitution preserves well-formedness of types). *If* $\Theta_L, \alpha, \Theta_R \vdash \mathsf{T}\ \text{type}^{\pm}$*, then* $\Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]\mathsf{T}\ \text{type}^{\pm}$.

Proof. By rule induction over the definition of well-formed types.

- **Case** $\dfrac{\beta \in \mathrm{FUV}(\Theta_L, \beta, \Theta_R)}{\Theta_L, \alpha, \Theta_R \vdash \alpha\ \text{type}^+}$ Twfuvar

  Take cases on whether $\beta = \alpha$

  - **Case** $\beta = \alpha$:

	$\Theta_L, \alpha, \Theta_R \vdash \alpha\ \text{type}^+$	Assumption
	$[\hat{\alpha}/\alpha]\alpha = \hat{\alpha}$	By definition
	$\Theta_L, \hat{\alpha}, \Theta_R \vdash \hat{\alpha}\ \text{type}^+$	By Twfguess
☞	$\Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]\beta\ \text{type}^+$	By equality

  - **Case** $\beta \neq \alpha$:

$\Theta_L, \alpha, \Theta_R \vdash \beta\ \text{type}^+$	Assumption
$[\hat{\alpha}/\alpha]\beta = \beta$	By definition

  Therefore, $\beta \in \mathrm{FUV}(\Theta_L)$ or $\beta \in \mathrm{FUV}(\Theta_R)$

	$\beta \in \mathrm{FUV}(\Theta_L, \hat{\alpha}, \Theta_R)$	
	$\Theta_L, \hat{\alpha}, \Theta_R \vdash \beta\ \text{type}^+$	By Twfuvar
☞	$\Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]\beta\ \text{type}^+$	By equality

- **Case** $\dfrac{\hat{\alpha} \in \mathrm{FEV}(\Theta_L, \alpha, \Theta_R)}{\Theta_L, \alpha, \Theta_R \vdash \hat{\alpha}\ \text{type}^+}$ Twfguess

$\Theta_L, \alpha, \Theta_R \vdash \hat{\beta}\ \text{type}^+$	Assumption
$[\hat{\alpha}/\alpha]\hat{\beta} = \hat{\beta}$	By definition

  Therefore, $\hat{\beta} \in \mathrm{FEV}(\Theta_L)$ or $\hat{\beta} \in \mathrm{FEV}(\Theta_R)$

	$\hat{\beta} \in \mathrm{FEV}(\Theta_L, \hat{\alpha}, \Theta_R)$	
	$\Theta_L, \hat{\alpha}, \Theta_R \vdash \hat{\beta}\ \text{type}^+$	By Twfguess
☞	$\Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]\hat{\beta}\ \text{type}^+$	By equality

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_R \vdash \downarrow N\ \mathsf{type}^+}$ Twfshift↓

$$\begin{array}{rll}
& \Theta_L, \alpha, \Theta_R \vdash \downarrow N\ \ \mathsf{type}^+ & \text{Assumption} \\
& \Theta_L, \alpha, \Theta_R \vdash N\ \ \mathsf{type}^- & \text{Premise} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]N\ \ \mathsf{type}^- & \text{By i.h.} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash \downarrow[\hat{\alpha}/\alpha]N\ \ \mathsf{type}^+ & \text{By Twfshift}\downarrow \\
☞ & \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]\downarrow N\ \ \mathsf{type}^+ & \text{By definition of substitution}
\end{array}$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R, \beta \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_R \vdash \forall\beta.N\ \mathsf{type}^-}$ Twfforall

  β fresh, and therefore $\beta \neq \alpha$.

$$\begin{array}{rll}
& \Theta_L, \alpha, \Theta_R \vdash \forall\beta.N\ \ \mathsf{type}^- & \text{Assumption} \\
& \Theta_L, \alpha, (\Theta_R, \beta) \vdash N\ \ \mathsf{type}^- & \text{Premise} \\
& \Theta_L, \hat{\alpha}, (\Theta_R, \beta) \vdash [\hat{\alpha}/\alpha]N\ \ \mathsf{type}^- & \text{By i.h.} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash \forall\beta.[\hat{\alpha}/\alpha]N\ \ \mathsf{type}^- & \text{By Twfforall} \\
☞ & \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha](\forall\beta.N)\ \ \mathsf{type}^- & \text{By definition of substitution}
\end{array}$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash P\ \mathsf{type}^+ \qquad \Theta_L, \alpha, \Theta_R \vdash N\ \mathsf{type}^-}{\Theta_L, \alpha, \Theta_R \vdash P \rightarrow N\ \mathsf{type}^-}$ Twfarrow

$$\begin{array}{rll}
& \Theta_L, \alpha, \Theta_R \vdash P \rightarrow N\ \ \mathsf{type}^- & \text{Assumption} \\
& \Theta_L, \alpha, \Theta_R \vdash P\ \ \mathsf{type}^+ & \text{Assumption} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]P\ \ \mathsf{type}^+ & \text{By i.h.} \\
& \Theta_L, \alpha, \Theta_R \vdash N\ \ \mathsf{type}^- & \text{Assumption} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]N\ \ \mathsf{type}^- & \text{By i.h.} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash ([\hat{\alpha}/\alpha]P) \rightarrow ([\hat{\alpha}/\alpha]N)\ \ \mathsf{type}^- & \text{By Twfarrow} \\
☞ & \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha](P \rightarrow N)\ \ \mathsf{type}^- & \text{By definition of substitution}
\end{array}$$

- **Case** $\dfrac{\Theta_L, \alpha, \Theta_R \vdash P\ \mathsf{type}^+}{\Theta_L, \alpha, \Theta_R \vdash \uparrow P\ \mathsf{type}^-}$ Twfshift↑

$$\begin{array}{rll}
& \Theta_L, \alpha, \Theta_R \vdash \uparrow P\ \ \mathsf{type}^- & \text{Assumption} \\
& \Theta_L, \alpha, \Theta_R \vdash P\ \ \mathsf{type}^+ & \text{Premise} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]P\ \ \mathsf{type}^+ & \text{By i.h.} \\
& \Theta_L, \hat{\alpha}, \Theta_R \vdash \uparrow[\hat{\alpha}/\alpha]P\ \ \mathsf{type}^- & \text{By Twfshift}\uparrow \\
☞ & \Theta_L, \hat{\alpha}, \Theta_R \vdash [\hat{\alpha}/\alpha]\uparrow P\ \ \mathsf{type}^- & \text{By definition of substitution}
\end{array}$$

□

Lemma K.4 (Context extension maintains variables). *If* $\Theta \longrightarrow \Omega$*, then* $\mathrm{FUV}(\Theta) = \mathrm{FUV}(\Omega)$ *and* $\mathrm{FEV}(\Theta) = \mathrm{FEV}(\Omega)$.

Proof. All rules ensure that the left-hand side and right-hand side contexts have the same set of free universal variables and the same set of existential variables. □

Lemma K.5 (Algorithmic typing is w.f.). *Given a typing context* Θ *and typing environment* Γ *such that* $\Theta\,\mathsf{ctx}$ *and* $\Theta \vdash \Gamma\,\mathsf{env}$*:*

- *If* $\Theta; \Gamma \vdash v : P \dashv \Theta'$*, then* $\Theta'\,\mathsf{ctx}$*,* $\Theta \longrightarrow \Theta'$*,* $\Theta' \vdash P\,\mathsf{type}^+$*, and* P *ground.*
- *If* $\Theta; \Gamma \vdash t : N \dashv \Theta'$*, then* $\Theta'\,\mathsf{ctx}$*,* $\Theta \longrightarrow \Theta'$*,* $\Theta' \vdash N\,\mathsf{type}^-$*, and* N *ground.*
- *If* $\Theta; \Gamma \vdash s : N \gg M \dashv \Theta'$*,* $\Theta \vdash N\,\mathsf{type}^-$*, and* $[\Theta]N = N$*, then* $\Theta'\,\mathsf{ctx}$*,* $\Theta \Longrightarrow \Theta'$*,* $\Theta' \vdash M\,\mathsf{type}^-$*,* $[\Theta']M = M$*, and* $\mathrm{FEV}(M) \subseteq \mathrm{FEV}(N) \cup (\mathrm{FEV}(\Theta') \setminus \mathrm{FEV}(\Theta))$.

Proof. By mutual rule induction over the algorithmic synthesis and spine judgments.

- **Case** $\dfrac{x : P \in \Gamma}{\Theta; \Gamma \vdash x : P \dashv \Theta}$ Avar

☞	$\Theta\ \mathsf{ctx}$	Assumption
☞	$\Theta \longrightarrow \Theta$	By Lemma D.1 (Context extension is reflexive)
	$\Theta \vdash \Gamma\,\mathsf{env}$	Assumption
☞	$\Theta \vdash P\ \mathsf{type}^+$	Inversion (Ewfvar)
☞	P ground	″

- **Case** $\dfrac{\Theta; \Gamma, x : P \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N \dashv \Theta'}$ Aλabs

	$\Theta\ \mathsf{ctx}$	Assumption
	$\Theta \vdash \Gamma\,\mathsf{env}$	Assumption
	$\Theta \vdash P\ \mathsf{type}^+$	P annotation
	P ground	″
	$\Theta \vdash \Gamma, x : P\,\mathsf{env}$	By Ewfvar
	$\Theta; \Gamma, x : P \vdash t : N \dashv \Theta'$	Subderivation
☞	$\Theta'\ \mathsf{ctx}$	By i.h.
☞	$\Theta \longrightarrow \Theta'$	″
	$\Theta' \vdash N\ \mathsf{type}^-$	″
	N ground	″
☞	$\Theta' \vdash P \to N\ \mathsf{type}^-$	By Twfarrow
☞	$P \to N$ ground	By definition of ground

- **Case** $$\frac{\Theta, \alpha; \Gamma \vdash t : N \dashv \Theta', \alpha}{\Theta; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N \dashv \Theta'}\ \mathsf{Agen}$$

$$\begin{array}{rll}
& \Theta\ \ \text{ctx} & \text{Assumption} \\
& \Theta, \alpha\ \ \text{ctx} & \text{By Cwfuvar} \\
& \Theta \vdash \Gamma\, \text{env} & \text{Assumption} \\
& \Theta, \alpha \vdash \Gamma\, \text{env} & \text{By weakening} \\
& \Theta, \alpha; \Gamma \vdash t : N \dashv \Theta', \alpha & \text{Subderivation} \\
& \Theta', \alpha\ \ \text{ctx} & \text{By i.h.} \\
& \Theta, \alpha \longrightarrow \Theta', \alpha & '' \\
& \Theta, \alpha \vdash N\ \ \text{type}^- & '' \\
& [\Theta']N\ \ \text{ground} & '' \\
☞ & \Theta'\ \ \text{ctx} & \text{Inversion (Cwfuvar)} \\
☞ & \Theta \longrightarrow \Theta' & \text{Inversion (Cuvar)} \\
☞ & \Theta \vdash \forall\alpha.\, N\ \ \text{type}^- & \text{By Twfforall} \\
☞ & [\Theta'](\forall\alpha.\, N)\ \ \text{ground} & \text{By definition of ground}
\end{array}$$

- **Case** $$\frac{\Theta; \Gamma \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \{t\} : {\downarrow}N \dashv \Theta'}\ \mathsf{Athunk}$$

$$\begin{array}{rll}
& \Theta\ \ \text{ctx} & \text{Assumption} \\
& \Theta \vdash \Gamma\, \text{env} & \text{Assumption} \\
& \Theta \vdash \Gamma\, \text{env} & \text{By Ewfvar} \\
& \Theta; \Gamma \vdash t : N \dashv \Theta' & \text{Subderivation} \\
☞ & \Theta'\ \ \text{ctx} & \text{By i.h.} \\
☞ & \Theta \longrightarrow \Theta' & '' \\
& \Theta' \vdash N\ \ \text{type}^- & '' \\
& [\Theta']N\ \ \text{ground} & '' \\
☞ & \Theta' \vdash {\downarrow}N\ \ \text{type}^- & \text{By } \leq^{\pm}\mathsf{Dshift}{\downarrow} \\
☞ & [\Theta']{\downarrow}N\ \ \text{ground} & \text{By definition of ground}
\end{array}$$

- **Case** $$\frac{\Theta; \Gamma \vdash v : P \dashv \Theta'}{\Theta; \Gamma \vdash \mathsf{return}\ v : {\uparrow}P \dashv \Theta'}\ \mathsf{Areturn}$$

  Symmetrical to Athunk.

- **Case** $$\frac{\begin{array}{c}\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta' \qquad \Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta'' \qquad \Theta'' \vdash P \leq^+ Q \dashv \Theta''' \\ \Theta''' \vdash [\Theta''']Q \leq^+ P \dashv \Theta^{(4)} \qquad \Theta^{(5)} = \Theta^{(4)} \restriction \Theta \qquad \Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta^{(6)}\end{array}}{\Theta; \Gamma \vdash \mathsf{let}\ x : P = v(s); t : N \dashv \Theta^{(6)}}\ \mathsf{Aambiguouslet}$$

Apply the induction hypothesis to the first premise:

	$\Theta \text{ ctx}$	Assumption
	$\Theta \vdash \Gamma \text{ env}$	Assumption
	$\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta'$	Subderivation
	$\Theta' \text{ ctx}$	By i.h.
(1)	$\Theta \longrightarrow \Theta'$	″
	$\Theta' \vdash {\downarrow}M \text{ type}^+$	″
	${\downarrow}M \text{ ground}$	″

Apply the induction hypothesis again, this time to the second premise:

	$\Theta \text{ ctx}$	Assumption
	$\Theta \Longrightarrow \Theta'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
	$\Theta' \vdash \Gamma \text{ env}$	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta''$	Subderivation
	$\Theta' \vdash M \text{ type}^-$	Inversion (Twfshift↓)
	$M \text{ ground}$	By definition of ground and above
	$[\Theta']M = M$	By Lemma D.5 (Applying a context to a ground type)
(2)	$\Theta' \Longrightarrow \Theta''$	By i.h.
	$\Theta'' \text{ ctx}$	″
	$\Theta'' \vdash {\uparrow}Q \text{ type}^-$	″
	$[\Theta'']{\uparrow}Q = {\uparrow}Q$	″

Now apply the well-formedness of algorithmic subtyping to the third premise:

	$\Theta'' \vdash P \leq^+ Q \dashv \Theta'''$	Subderivation
	$\Theta'' \text{ ctx}$	Above
	$P \text{ ground}$	P annotation
	$[\Theta'']Q = Q$	By definition of $[-]-$ and above
	$\Theta''' \text{ ctx}$	By Lemma E.2 (Algorithmic subtyping is w.f.)
(3)	$\Theta'' \longrightarrow \Theta'''$	″
	$[\Theta''']Q \text{ ground}$	″

Apply it again to the fourth premise:

	$\Theta''' \vdash [\Theta''']Q \leq^+ P \dashv \Theta^{(4)}$	Subderivation
	$\Theta''' \text{ ctx}$	Above
	$[\Theta''']Q \text{ ground}$	Above
	$[\Theta''']P = P$	By Lemma D.5 (Applying a context to a ground type)
	$\Theta^{(4)} \text{ ctx}$	By Lemma E.2 (Algorithmic subtyping is w.f.)
(4)	$\Theta''' \longrightarrow \Theta^{(4)}$	″

Make use of Lemma K.1 (Well-formedness of restricted contexts) in the context of the fifth premise:

$\Theta \Longrightarrow \Theta'$	Applying Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$) to (1)
$\Theta' \Longrightarrow \Theta''$	Above ((2))
$\Theta'' \Longrightarrow \Theta'''$	Applying

		Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$) to (3)
	$\Theta''' \Longrightarrow \Theta^{(4)}$	Applying Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$) to (4)
	Θ ctx	Above
	$\Theta^{(4)}$ ctx	Above
	$\Theta \Longrightarrow \Theta^{(4)}$	By Lemma C.4 (Weak context extension is transitive)
	$\Theta^{(5)} = \Theta^{(4)} \restriction \Theta$	By premise
(5)	$\Theta \longrightarrow \Theta^{(5)}$	By Lemma K.1 (Well-formedness of restricted contexts)
	$\Theta^{(5)}$ ctx	″

Finally, apply the induction hypothesis to the last premise:

	$\Theta^{(5)}$ ctx	Above
	$\Theta \Longrightarrow \Theta^{(5)}$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
	$\Theta^{(5)} \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta \vdash P$ type$^+$	P is an annotation
	$\Theta^{(5)} \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
	P ground	P is an annotation
	$\Theta^{(5)} \vdash \Gamma, x : P$ env	By Ewfvar
	$\Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta^{(6)}$	Subderivation
☞	$\Theta^{(6)}$ ctx	By i.h.
(6)	$\Theta^{(5)} \longrightarrow \Theta^{(6)}$	″
☞	$\Theta^{(6)} \vdash N$ type$^-$	″
☞	N ground	″
☞	$\Theta \longrightarrow \Theta^{(6)}$	Applying Lemma D.3 (Context extension is transitive) to (5) and (6)

- **Case**

$$\dfrac{\begin{array}{c}\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta' \\ \Theta'; \Gamma \vdash s : M \gg \uparrow Q \dashv \Theta'' \qquad \mathrm{FEV}(Q) = \emptyset \qquad \Theta''' = \Theta'' \restriction \Theta \qquad \Theta'''; \Gamma, x : Q \vdash t : N \dashv \Theta^{(4)}\end{array}}{\Theta; \Gamma \vdash \mathsf{let}\ x = v(s); t : N \dashv \Theta^{(4)}}\ \textsf{Aunambiguouslet}$$

First apply the induction hypothesis to the first subderivation:

	Θ ctx	Assumption
	$\Theta \vdash \Gamma$ env	Assumption
	$\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta'$	Subderivation
	Θ' ctx	By i.h.
(1)	$\Theta \longrightarrow \Theta'$	″
	$\Theta' \vdash \downarrow M$ type$^+$	″
	$\downarrow M$ ground	″

Now apply the induction hypothesis to the spine subderivation:

	Θ' ctx	Above

	$\Theta \Longrightarrow \Theta'$	Applying Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$) to (1)
	$\Theta' \vdash \Gamma\ \mathsf{env}$	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta''$	Subderivation
	M ground	By the definition of ground
	$\Theta' \vdash M\ \mathsf{type}^-$	Inversion (Twfshift↓)
	$[\Theta']M = M$	By Lemma D.5 (Applying a context to a ground type)
	Θ'' ctx	By i.h.
	$\Theta'' \vdash {\uparrow}Q\ \mathsf{type}^-$	″
	$\Theta' \Longrightarrow \Theta''$	″

Produce a strong context extension judgment using Lemma K.1 (Well-formedness of restricted contexts):

	Θ ctx	Above
	Θ'' ctx	Above
	$\Theta \Longrightarrow \Theta''$	By Lemma C.4 (Weak context extension is transitive)
	$\Theta''' = \Theta'' \upharpoonright \Theta$	Premise
(2)	$\Theta \longrightarrow \Theta'''$	By Lemma K.1 (Well-formedness of restricted contexts)
	Θ''' ctx	″

Finally, apply the induction hypothesis to the last premise:

	$\Theta \Longrightarrow \Theta'''$	Applying Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$) to (2)
	$\Theta''' \vdash \Gamma\ \mathsf{env}$	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta'' \vdash Q\ \mathsf{type}^+$	Inversion (Twfshift↑)
	$\Theta''' \vdash Q\ \mathsf{type}^+$	By Lemma D.4 (Context extension preserves w.f.)
	$\mathrm{FEV}(Q) = \emptyset$	Premise
	Q ground	By definition of ground
	$\Theta''' \vdash \Gamma, x : Q\ \mathsf{env}$	By Ewfvar
	Θ''' ctx	Above
	$\Theta''' \vdash \Gamma, x : Q\ \mathsf{env}$	Above
	$\Theta'''; \Gamma, x : Q \vdash t : N \dashv \Theta^{(4)}$	Subderivation
☞	$\Theta^{(4)}$ ctx	By i.h.
(3)	$\Theta''' \longrightarrow \Theta^{(4)}$	″
☞	$\Theta^{(4)} \vdash N\ \mathsf{type}^-$	″
☞	N ground	″
☞	$\Theta \longrightarrow \Theta^{(4)}$	Applying Lemma D.3 (Context extension is transitive) to (2) and (3)

- **Case**

$$\frac{}{\Theta; \Gamma \vdash \epsilon : N \gg N \dashv \Theta}\ \mathsf{Aspinenil}$$

☞	Θ ctx	Assumption
☞	$\Theta \Longrightarrow \Theta$	By Lemma C.2 (Weak context extension is reflexive)
☞	$\Theta \vdash N\ \mathsf{type}^-$	Assumption
☞	$[\Theta]N = N$	Assumption
	$\mathrm{FEV}(N) \subseteq \mathrm{FEV}(N)$	By reflexivity of $\subseteq$

☞ $\mathsf{FEV}(N) \subseteq \mathsf{FEV}(N) \cup (\mathsf{FEV}(\Theta') \setminus \mathsf{FEV}(\Theta))$ By definition of $\subseteq$

- **Case** $$\dfrac{\Theta; \Gamma \vdash v : P \dashv \Theta' \qquad \Theta' \vdash P \leq^{+} [\Theta']Q \dashv \Theta'' \qquad \Theta''; \Gamma \vdash s : [\Theta'']N \gg M \dashv \Theta'''}{\Theta; \Gamma \vdash v, s : Q \rightarrow N \gg M \dashv \Theta'''}\ \textsf{Aspinecons}$$

Θ ctx	Assumption
$\Theta \vdash \Gamma$ env	Assumption
$\Theta; \Gamma \vdash v : P \dashv \Theta'$	Subderivation
Θ' ctx	By i.h.
$\Theta \longrightarrow \Theta'$	"
$\Theta' \vdash P$ type^{+}	"
P ground	"
$\Theta' \vdash P \leq^{+} Q \dashv \Theta''$	Subderivation
Θ' ctx	Above
P ground	Above
$[\Theta'][\Theta']Q = [\Theta']Q$	By Lemma D.6 (Context application is idempotent)
Θ'' ctx	By Lemma E.2 (Algorithmic subtyping is w.f.)
$\Theta' \longrightarrow \Theta''$	"
$[\Theta''][\Theta']Q$ ground	"
Θ'' ctx	Above
$\Theta \longrightarrow \Theta''$	By Lemma B.7 (Declarative subtyping is transitive)
$\Theta \Longrightarrow \Theta''$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
$\Theta'' \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta''; \Gamma \vdash s : [\Theta'']N \gg \uparrow Q \dashv \Theta'''$	Subderivation
$\Theta \vdash N$ type^{-}	Inversion (Twfarrow)
$\Theta'' \vdash N$ type^{-}	By Lemma D.4 (Context extension preserves w.f.)
$\Theta'' \vdash [\Theta'']N$ type^{-}	By Lemma E.1 (Applying context to the type preserves w.f.)
$[\Theta''][\Theta'']N = [\Theta'']N$	By Lemma D.6 (Context application is idempotent)
☞ Θ''' ctx	By i.h.
$\Theta'' \Longrightarrow \Theta'''$	"
☞ $\Theta''' \vdash M$ type^{-}	"
☞ $[\Theta''']M = M$	"
$\mathsf{FEV}(M) \subseteq \mathsf{FEV}(N) \cup (\mathsf{FEV}(\Theta''') \setminus \mathsf{FEV}(\Theta''))$	"
☞ $\Theta \Longrightarrow \Theta'''$	By Lemma C.4 (Weak context extension is transitive)
$\mathsf{FEV}(N) \subseteq \mathsf{FEV}(Q \rightarrow N)$	By definition of FEV
$\mathsf{FEV}(\Theta) = \mathsf{FEV}(\Theta'')$	By Lemma K.4 (Context extension maintains variables)
☞ $\mathsf{FEV}(M) \subseteq \mathsf{FEV}(Q \rightarrow N) \cup (\mathsf{FEV}(\Theta''') \setminus \mathsf{FEV}(\Theta))$	Substituting above

- **Case** $$\dfrac{\Theta; \Gamma \vdash s : N \gg M \dashv \Theta' \qquad \alpha \notin \mathsf{FUV}(N)}{\Theta; \Gamma \vdash s : (\forall \alpha.\, N) \gg M \dashv \Theta'}\ \textsf{Aspinetypeabsnotin}$$

	Θ ctx	Assumption
	$\Theta \vdash \Gamma$ env	Assumption
	$\Theta; \Gamma \vdash s : N \gg M \dashv \Theta'$	Subderivation
	$\Theta \vdash \forall\alpha.\, N$ type$^-$	Assumption
	$\Theta, \alpha \vdash N$ type$^-$	Inversion (Twfforall)
	$\Theta \vdash N$ type$^-$	By Lemma K.2 (Type well-formed with type variable removed)
	$[\Theta](\forall\alpha.\, N) = \forall\alpha.\, N$	Assumption
	$\forall\alpha.\, [\Theta]N = \forall\alpha.\, N$	By definition of $[-]-$
	$[\Theta]N = N$	By equality
☞	Θ' ctx	By i.h.
☞	$\Theta \Longrightarrow \Theta'$	"
☞	$\Theta' \vdash M$ type$^-$	"
☞	$[\Theta']M = M$	"
	$FEV(M) \subseteq FEV(N) \cup (FEV(\Theta') \setminus FEV(\Theta))$	"
	$FEV(\forall\alpha.\, N) = FEV(N)$	By definition of FEV
☞	$FEV(M) \subseteq FEV(\forall\alpha.\, N) \cup (FEV(\Theta') \setminus FEV(\Theta))$	Substituting above

- **Case** $\dfrac{\Theta, \hat\alpha; \Gamma \vdash s : [\hat\alpha/\alpha]N \gg M \dashv \Theta', \hat\alpha\,[= P] \qquad \alpha \in FUV(N)}{\Theta; \Gamma \vdash s : (\forall\alpha.\, N) \gg M \dashv \Theta', \hat\alpha\,[= P]}$ Aspinetypeabsin

Θ ctx	Assumption
$\Theta, \hat\alpha$ ctx	By Cwfunsolvedguess
$\Theta \vdash \Gamma$ env	Assumption
$\Theta \Longrightarrow \Theta, \hat\alpha$	By Lemma C.2 (Weak context extension is reflexive) and Wcnewunsolvedguess
$\Theta, \hat\alpha \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta, \hat\alpha; \Gamma \vdash s : [\hat\alpha/\alpha]N \gg M \dashv \Theta', \hat\alpha\,[= P]$	Subderivation
$\Theta \vdash \forall\alpha.\, N$ type$^-$	Assumption
$\alpha \notin FUV(\Theta)$	α fresh
$\Theta, \alpha \vdash N$ type$^-$	By Twfforall
$\Theta, \hat\alpha \vdash [\hat\alpha/\alpha]N$ type$^-$	By Lemma K.3 (Substitution preserves well-formedness of types)
$[\Theta](\forall\alpha.\, N) = \forall\alpha.\, N$	Assumption
$[\Theta]N = N$	By definition of $[-]-$
$[\Theta]([\hat\alpha/\alpha]N) = [\hat\alpha/\alpha]N$	$\hat\alpha$ fresh
$[\Theta, \hat\alpha]([\hat\alpha/\alpha]N) = [\hat\alpha/\alpha]N$	By definition of $[-]-$

☞	$\Theta', \hat\alpha\,[= P]$ ctx	By i.h.
	$\Theta, \hat\alpha \Longrightarrow \Theta', \hat\alpha\,[= P]$	"
☞	$\Theta', \hat\alpha\,[= P] \vdash M$ type$^-$	"
☞	$[\Theta', \hat\alpha\,[= P]]M = M$	"
	$FEV(M) \subseteq FEV([\hat\alpha/\alpha]N) \cup (FEV(\Theta', \hat\alpha\,[= P]) \setminus FEV(\Theta, \hat\alpha))$	"

$\Theta \Longrightarrow \Theta$ By Lemma C.2 (Weak context extension is reflexive)
$\Theta \Longrightarrow \Theta, \hat{\alpha}$ By $\hat{\alpha}$ fresh & Wcnewunsolvedguess
☞ $\Theta \Longrightarrow \Theta', \hat{\alpha}\,[= P]$ By Lemma C.4 (Weak context extension is transitive)

$\text{FEV}(M) \subseteq \text{FEV}([\hat{\alpha}/\alpha]N) \cup (\text{FEV}(\Theta', \hat{\alpha}\,[= P]) \setminus \text{FEV}(\Theta))$ By definition of FEV
$\text{FEV}([\hat{\alpha}/\alpha]N) \subseteq \text{FEV}(\forall\alpha.\, N) \cup \{\hat{\alpha}\}$ By definition of FEV
$\{\hat{\alpha}\} \subseteq \text{FEV}(\Theta', \hat{\alpha}\,[= P]) \setminus \text{FEV}(\Theta)$ By definition of FEV
☞ $\text{FEV}(M) \subseteq \text{FEV}(\forall\alpha.\, N) \cup (\text{FEV}(\Theta', \hat{\alpha}\,[= P]) \setminus \text{FEV}(\Theta))$ By above

□

L' Determinism of typing

Lemma L.1 (Algorithmic typing is deterministic)**.**

- *If* $\Theta; \Gamma \vdash e : A_1 \dashv \Theta'_1$ *and* $\Theta; \Gamma \vdash e : A_2 \dashv \Theta'_2$, *then* $A_1 = A_2$ *and* $\Theta'_1 = \Theta'_2$.
- *If* $\Theta; \Gamma \vdash t : N \gg M_1 \dashv \Theta'_1$ *and* $\Theta; \Gamma \vdash t : N \gg M_2 \dashv \Theta'_2$, *then* $M_1 = M_2$ *and* $\Theta'_1 = \Theta'_2$.

Proof. The algorithmic system is mostly syntax-oriented, with the only exceptions Aspinetypeabsnotin and Aspinetypeabsin (which have the same conclusion) being distinguished by whether $\alpha \in \text{FUV}(N)$, a deterministic check. Therefore, determinacy of the system follows by a straightforward mutual rule induction over the algorithmic synthesis and spine judgments, making use of Lemma H.1 (Algorithmic subtyping is deterministic). □

M' Decidability of typing

Lemma M.1 (Decidability of algorithmic typing)**.** *There exists a total order* $\sqsubset$ *on well-formed algorithmic typing judgments such that for each derivation with typing judgment premises* A_i *and conclusion* B*, each* A_i *compares less than* B*, i.e.* $\forall i.\, A_i \sqsubset B$.

Proof. We use the same ordering of judgments as in Lemma J.1 (Isomorphic environments type the same terms).

- **Case** $$\frac{x : P \in \Gamma}{\Theta; \Gamma \vdash x : P \dashv \Theta}\ \text{Avar}$$

  Testing membership of Γ terminates since typing environments are finite.

- **Case** $$\frac{\Theta; \Gamma, x : P \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N \dashv \Theta'}\ \text{A}\lambda\text{abs}$$

$|\lambda x : P.\, t| = |t| + 1$ By definition of $|\_|$
$> |t|$
☞ $(\Theta; \Gamma, x : P \vdash t : N \dashv \Theta')$
$\sqsubset$ By definition of $\sqsubset$
$(\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N \dashv \Theta')$

- **Case** $$\frac{\Theta, \alpha; \Gamma \vdash t : N \dashv \Theta', \alpha}{\Theta; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N \dashv \Theta'}\ \textsf{Agen}$$

$$\begin{array}{lll} & |\Lambda\alpha.\, t| = |t| + 1 & \text{By definition of } |\_| \\ & \quad > |t| & \\ ☞ & (\Theta, \alpha; \Gamma \vdash t : N \dashv \Theta', \alpha) & \\ & \sqsubset & \text{By definition of } \sqsubset \\ & (\Theta; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N \dashv \Theta') & \end{array}$$

- **Case** $$\frac{\Theta; \Gamma \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \{t\} : {\downarrow}N \dashv \Theta'}\ \textsf{Athunk}$$

$$\begin{array}{lll} & |\{t\}| = |t| + 1 & \text{By definition of } |\_| \\ & \quad > |t| & \\ ☞ & (\Theta; \Gamma \vdash t : N \dashv \Theta') & \\ & \sqsubset & \text{By definition of } \sqsubset \\ & (\Theta; \Gamma \vdash \{t\} : {\downarrow}N \dashv \Theta') & \end{array}$$

- **Case** $$\frac{\Theta; \Gamma \vdash v : P \dashv \Theta'}{\Theta; \Gamma \vdash \mathsf{return}\ v : {\uparrow}P \dashv \Theta'}\ \textsf{Areturn}$$

$$\begin{array}{lll} & |\mathsf{return}\ v| = |v| + 1 & \text{By definition of } |\_| \\ & \quad > |v| & \\ ☞ & (\Theta; \Gamma \vdash v : P \dashv \Theta') & \\ & \sqsubset & \text{By definition of } \sqsubset \\ & (\Theta; \Gamma \vdash \mathsf{return}\ v : {\uparrow}P \dashv \Theta') & \end{array}$$

- **Case** $$\frac{\begin{array}{ccc}\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta' & \Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta'' & \Theta'' \vdash P \leq^{+} Q \dashv \Theta''' \\ \Theta''' \vdash [\Theta''']Q \leq^{+} P \dashv \Theta^{(4)} & \Theta^{(5)} = \Theta^{(4)} \upharpoonright \Theta & \Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta^{(6)}\end{array}}{\Theta; \Gamma \vdash \mathsf{let}\ x : P = v(s); t : N \dashv \Theta^{(6)}}\ \textsf{Aambiguouslet}$$

The algorithmic subtyping judgments terminate per Lemma I.2 (Decidability of algorithmic subtyping).

$$\begin{array}{lll} & |\mathsf{let}\ x : P = v(s); t| = |v| + |s| + |t| + 1 & \text{By definition of } |\_| \\ & \quad |v| < |\mathsf{let}\ x : P = v(s); t| & \\ ☞ & (\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta') & \\ & \sqsubset & \text{By definition of } \sqsubset \\ & (\Theta; \Gamma \vdash \mathsf{let}\ x : P = v(s); t : N \dashv \Theta^{(6)}) & \\ & & \\ & \quad |s| < |\mathsf{let}\ x : P = v(s); t| & \\ ☞ & (\Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta'') & \\ & \sqsubset & \text{By definition of } \sqsubset \end{array}$$

$(\Theta; \Gamma \vdash \text{let } x : P = v(s); t : N \dashv \Theta^{(6)})$

☞ $|t| < |\text{let } x : P = v(s); t|$
$(\Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta^{(6)})$
$\sqcap$ By definition of $\sqsubset$
$(\Theta; \Gamma \vdash \text{let } x : P = v(s); t : N \dashv \Theta^{(6)})$

- **Case**

$$\frac{\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta' \quad \Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta'' \quad \text{FEV}(Q) = \emptyset \quad \Theta''' = \Theta'' \upharpoonright \Theta \quad \Theta'''; \Gamma, x : Q \vdash t : N \dashv \Theta^{(4)}}{\Theta; \Gamma \vdash \text{let } x = v(s); t : N \dashv \Theta^{(4)}} \text{Aunambiguouslet}$$

Determining the set of free universal variables of a finite type is terminating.

$|\text{let } x = v(s); t| = |v| + |s| + |t| + 1$ By definition of $|\_|$
$|v| < |\text{let } x = v(s); t|$
☞ $(\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta')$
$\sqcap$ By definition of $\sqsubset$
$(\Theta; \Gamma \vdash \text{let } x = v(s); t : N \dashv \Theta^{(4)})$

$|s| < |\text{let } x = v(s); t|$
☞ $(\Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta'')$
$\sqcap$ By definition of $\sqsubset$
$(\Theta; \Gamma \vdash \text{let } x = v(s); t : N \dashv \Theta^{(4)})$

$|t| < |\text{let } x = v(s); t|$
☞ $(\Theta'''; \Gamma, x : P \vdash t : N \dashv \Theta^{(4)})$
$\sqcap$ By definition of $\sqsubset$
$(\Theta; \Gamma \vdash \text{let } x = v(s); t : N \dashv \Theta^{(4)})$

- **Case**

$$\frac{}{\Theta; \Gamma \vdash \epsilon : N \gg N \dashv \Theta} \text{Aspinenil}$$

Rule terminal.

- **Case**

$$\frac{\Theta; \Gamma \vdash v : P \dashv \Theta' \quad \Theta' \vdash P \leq^{+} [\Theta']Q \dashv \Theta'' \quad \Theta''; \Gamma \vdash s : [\Theta'']N \gg M \dashv \Theta'''}{\Theta; \Gamma \vdash v, s : Q \to N \gg M \dashv \Theta'''} \text{Aspinecons}$$

The algorithmic subtyping judgment terminates per Lemma I.2 (Decidability of algorithmic subtyping).

$|v, s| = |v| + |s| + 1$ By definition of $|\_|$
$> |v|$
☞ $(\Theta; \Gamma \vdash v : P \dashv \Theta')$
$\sqcap$ By definition of $\sqsubset$
$(\Theta; \Gamma \vdash v, s : Q \to N \gg M \dashv \Theta''')$
$|v, s| > |s|$
☞ $(\Theta''; \Gamma \vdash s : [\Theta'']N \gg M \dashv \Theta''')$
By definition of $\sqsubset$

$$\sqcap$$
$$\Theta; \Gamma \vdash v, s : Q \to N \gg M \dashv \Theta'''$$

- **Case** $$\frac{\Theta; \Gamma \vdash s : N \gg M \dashv \Theta' \qquad \alpha \notin FUV(N)}{\Theta; \Gamma \vdash s : (\forall \alpha.\, N) \gg M \dashv \Theta'} \text{Aspinetypeabsnotin}$$

$$|s| = |s|$$
$$NPQ(\forall \alpha.\, N) > NPQ(N)$$

☞ $(\Theta; \Gamma \vdash s : N \gg M \dashv \Theta')$ $\sqcap$ $(\Theta; \Gamma \vdash s : (\forall \alpha.\, N) \gg M \dashv \Theta')$ — By definition of $\sqsubset$

We define types to be finite, therefore calculating $FUV(N)$ is terminating.

- **Case** $$\frac{\Theta, \hat{\alpha}; \Gamma \vdash s : [\hat{\alpha}/\alpha]N \gg M \dashv \Theta', \hat{\alpha}\,[= P] \qquad \alpha \in FUV(N)}{\Theta; \Gamma \vdash s : (\forall \alpha.\, N) \gg M \dashv \Theta', \hat{\alpha}\,[= P]} \text{Aspinetypeabsin}$$

$$|s| = |s|$$
$NPQ(\forall \alpha.\, N) > NPQ([\hat{\alpha}/\alpha]N)$ — Since α and $\hat{\alpha}$ are positive, the substitution cannot introduce any prenex quantifiers.

☞ $(\Theta, \hat{\alpha}; \Gamma \vdash s : [\hat{\alpha}/\alpha]N \gg M \dashv \Theta', \hat{\alpha}[= P])$ $\sqcap$ $(\Theta; \Gamma \vdash s : (\forall \alpha.\, N) \gg M \dashv \Theta')$ — By definition of $\sqsubset$

We define types to be finite, therefore calculating $FUV(N)$ is terminating.

□

N' Soundness of typing

N'.1 Lemmas

Lemma N.1 (Extended complete context). *If* Θ' ctx, Ω ctx, $\Theta \longrightarrow \Omega$, $\Theta \Longrightarrow \Theta'$, *and* $\Theta' \upharpoonright \Theta \longrightarrow \Omega$, *then* $\exists \Omega'$ *such that* Ω' ctx, $\Theta' \longrightarrow \Omega'$, *and* $\Omega \Longrightarrow \Omega'$.

Proof. We add the $\hat{\alpha}\,[= P]$ context items that newly appear in Θ' to the complete context.
By rule induction on $\Theta \Longrightarrow \Theta'$:

- **Case** $$\frac{}{\cdot \Longrightarrow \cdot} \text{Wcempty}$$

The new context is $\cdot$.

☞ $\cdot$ ctx — By Cwfempty
☞ $\cdot \longrightarrow \cdot$ — By Cempty
☞ $\cdot \Longrightarrow \cdot$ — By Wcempty

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \alpha \Longrightarrow \Theta', \alpha}$ Wcuvar

  We add the α context item onto the new context from the induction hypothesis.

	Θ', α ctx	Assumption
(1)	Θ' ctx	Inversion (Cwfuvar)
	$\Theta, \alpha \longrightarrow \Omega$	Assumption
	$\Omega = \bar{\Omega}, \alpha$	Inversion (Cuvar)
(2)	$\Theta \longrightarrow \bar{\Omega}$	$''$
	$\bar{\Omega}, \alpha$ ctx	Assumption
(3)	$\bar{\Omega}$ ctx	Inversion (Cwfuvar)
(4)	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta', \alpha \upharpoonright \Theta, \alpha \longrightarrow \bar{\Omega}, \alpha$	Assumption
(5)	$\Theta' \upharpoonright \Theta \longrightarrow \bar{\Omega}$	Inversion ($\upharpoonright$uvar)
	$\bar{\Omega}'$ ctx	By i.h., using (1–5) and for some complete context $\bar{\Omega}'$
	$\Theta' \longrightarrow \bar{\Omega}'$	$''$
	$\bar{\Omega} \Longrightarrow \bar{\Omega}'$	$''$
☞	$\bar{\Omega}', \alpha$ ctx	By Cwfuvar
☞	$\Theta', \alpha \longrightarrow \bar{\Omega}', \alpha$	By Cuvar
☞	$\bar{\Omega}, \alpha \Longrightarrow \bar{\Omega}', \alpha$	By Wcuvar

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha}}$ Wcunsolvedguess

  We add the $\hat{\alpha} = P$ context item to the new context from the induction hypothesis, where P is the solution for $\hat{\alpha}$ in the complete context Ω.

	$\Theta', \hat{\alpha}$ ctx	Assumption
(1)	Θ' ctx	Inversion (Cwfunsolvedguess)
	$\Theta, \hat{\alpha} \longrightarrow \Omega$	Assumption
	$\Theta, \hat{\alpha} \longrightarrow \Omega$	Assumption
	$\Omega = \bar{\Omega}, \hat{\alpha} = P$	Inversion (Csolveguess)
(2)	$\Theta \longrightarrow \bar{\Omega}$	$''$
	$\bar{\Omega}, \hat{\alpha} = P$ ctx	Assumption
(3)	$\bar{\Omega}$ ctx	Inversion (Cwfsolvedguess)
	$\bar{\Omega} \vdash P$ type$^+$	$''$
	P ground	$''$
(4)	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta', \hat{\alpha} \upharpoonright \Theta, \hat{\alpha} \longrightarrow \bar{\Omega}, \hat{\alpha} = P$	Assumption
	$(\Theta' \upharpoonright \Theta), \hat{\alpha} \longrightarrow \bar{\Omega}, \hat{\alpha} = P$	Inversion ($\upharpoonright$guessin)
(5)	$\Theta' \upharpoonright \Theta \longrightarrow \bar{\Omega}$	Inversion (Csolveguess)
	$\bar{\Omega}'$ ctx	By i.h., using (1–5) and for some complete context $\bar{\Omega}'$
	$\Theta' \longrightarrow \bar{\Omega}'$	$''$

$$\bar{\Omega} \Longrightarrow \bar{\Omega}' \qquad ''$$

$$\bar{\Omega}' \vdash P \;\; \mathsf{type}^+ \qquad \text{By Lemma C.5 (Weak context extension preserves well-formedness)}$$

☞ $\bar{\Omega}', \hat{\alpha} = P \;\; \mathsf{ctx}$ — By Cwfsolvedguess

☞ $\Theta', \hat{\alpha} \longrightarrow \bar{\Omega}', \hat{\alpha} = P$ — By Csolveguess

$[\Omega]\bar{\Omega} \vdash P \cong^+ P$ — By Lemma B.1 (Declarative subtyping is reflexive)

☞ $\bar{\Omega}, \hat{\alpha} = P \Longrightarrow \bar{\Omega}', \hat{\alpha} = P$ — By Wcsolvedguess

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta, \hat{\alpha} \Longrightarrow \Theta', \hat{\alpha} = P}$ Wcsolveguess

  As before, we add the $\hat{\alpha} = Q$ context item to the new context from the induction hypothesis, where Q is the solution for $\hat{\alpha}$ in the complete context Ω.

	$\Theta', \hat{\alpha} = P \;\; \mathsf{ctx}$	Assumption
(1)	$\Theta' \;\; \mathsf{ctx}$	Inversion (Cwfsolvedguess)
	$\Theta, \hat{\alpha} \longrightarrow \Omega$	Assumption
	$\Omega = \bar{\Omega}, \hat{\alpha} = Q$	Inversion (Csolveguess)
(2)	$\Theta \longrightarrow \bar{\Omega}$	$''$
	$\bar{\Omega}, \hat{\alpha} = Q \;\; \mathsf{ctx}$	Assumption
(3)	$\bar{\Omega} \;\; \mathsf{ctx}$	Inversion (Cwfsolvedguess)
	$\bar{\Omega} \vdash Q \;\; \mathsf{type}^+$	$''$
	$Q \;\; \mathsf{ground}$	$''$
(4)	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta', \hat{\alpha} = P \upharpoonright \Theta, \hat{\alpha} \longrightarrow \bar{\Omega}, \hat{\alpha} = Q$	Assumption
	$(\Theta' \upharpoonright \Theta), \hat{\alpha} = P \longrightarrow \bar{\Omega}, \hat{\alpha} = Q$	Inversion ($\upharpoonright$guessin)
(5)	$\Theta' \upharpoonright \Theta \longrightarrow \bar{\Omega}$	Inversion (Csolvedguess)
	$[\Omega]\Theta' \upharpoonright \Theta \vdash P \cong^+ Q$	$''$

$\bar{\Omega}' \;\; \mathsf{ctx}$	By i.h., using (1–5) and for some complete context $\bar{\Omega}'$
$\Theta' \longrightarrow \bar{\Omega}'$	$''$
$\bar{\Omega} \Longrightarrow \bar{\Omega}'$	$''$

$\bar{\Omega}' \vdash Q \;\; \mathsf{type}^+$ — By Lemma C.5 (Weak context extension preserves well-formedness)

☞ $\bar{\Omega}', \hat{\alpha} = Q \;\; \mathsf{ctx}$ — By Cwfsolvedguess

$\|\Theta'\| \vdash P \cong^+ Q$ — Since $[\Omega](\Theta' \upharpoonright \Theta) = [\Omega]\Theta'$

☞ $\Theta', \hat{\alpha} = P \longrightarrow \bar{\Omega}', \hat{\alpha} = Q$ — By Csolvedguess

$[\Omega]\bar{\Omega} \vdash Q \cong^+ Q$ — By Lemma B.1 (Declarative subtyping is reflexive)

☞ $\bar{\Omega}, \hat{\alpha} = Q \Longrightarrow \bar{\Omega}', \hat{\alpha} = Q$ — By Wcsolvedguess

- **Case** $\dfrac{\Theta \Longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^+ Q}{\Theta, \hat{\alpha} = P \Longrightarrow \Theta', \hat{\alpha} = Q}$ Wcsolvedguess

  We add the $\hat{\alpha} = R$ context item to the new context from the induction hypothesis, where R is the solution for $\hat{\alpha}$ in the complete context Ω.

	$\Theta', \hat{\alpha} = Q \;\; \mathsf{ctx}$	Assumption
(1)	$\Theta' \;\; \mathsf{ctx}$	Inversion (Cwfsolvedguess)
	$\Theta, \hat{\alpha} \longrightarrow \Omega$	Assumption
	$\Theta, \hat{\alpha} = P \longrightarrow \Omega$	Assumption
	$\Omega = \bar{\Omega}, \hat{\alpha} = R$	Inversion (Csolvedguess)
(2)	$\Theta \longrightarrow \bar{\Omega}$	"
	$\|\Theta\| \vdash P \cong^{+} R$	"
	$\bar{\Omega}, \hat{\alpha} = R \;\; \mathsf{ctx}$	Assumption
(3)	$\bar{\Omega} \;\; \mathsf{ctx}$	Inversion (Cwfsolvedguess)
	$\bar{\Omega} \vdash R \;\; \mathsf{type}^{+}$	"
	$R \;\; \mathsf{ground}$	"
(4)	$\Theta \Longrightarrow \Theta'$	Subderivation
	$\Theta', \hat{\alpha} = Q \upharpoonright \Theta, \hat{\alpha} = P \longrightarrow \bar{\Omega}, \hat{\alpha} = R$	Assumption
	$(\Theta' \upharpoonright \Theta), \hat{\alpha} = Q \longrightarrow \bar{\Omega}, \hat{\alpha} = R$	Inversion ($\upharpoonright$guessin)
(5)	$\Theta' \upharpoonright \Theta \longrightarrow \bar{\Omega}$	Inversion (Csolvedguess)
	$\bar{\Omega}' \;\; \mathsf{ctx}$	By i.h., using (1–5) and for some complete context $\bar{\Omega}'$
	$\Theta' \longrightarrow \bar{\Omega}'$	"
	$\bar{\Omega} \Longrightarrow \bar{\Omega}'$	"
	$\bar{\Omega}' \vdash R \;\; \mathsf{type}^{+}$	By Lemma C.5 (Weak context extension preserves well-formedness)
☞	$\bar{\Omega}', \hat{\alpha} = R \;\; \mathsf{ctx}$	By Cwfsolvedguess
	$\|\Theta\| \vdash P \cong^{+} Q$	Premise
	$\|\Theta\| \vdash Q \cong^{+} R$	By Lemma B.7 (Declarative subtyping is transitive)
	$\|\Theta'\| \vdash Q \cong^{+} R$	By Lemma C.3 (Equality of declarative contexts (weak))
☞	$\Theta', \hat{\alpha} = Q \longrightarrow \bar{\Omega}', \hat{\alpha} = R$	By Csolvedguess
	$[\Omega]\bar{\Omega} \vdash P \cong^{+} R$	By Lemma D.2 (Equality of declarative contexts)
☞	$\bar{\Omega}, \hat{\alpha} = P \Longrightarrow \bar{\Omega}', \hat{\alpha} = R$	By Wcsolvedguess

- **Case** $\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha}}$ Wcnewunsolvedguess

  We add a solved context item for $\hat{\alpha}$ to the new complete context from the induction hypothesis.

	$\Theta', \alpha \;\; \mathsf{ctx}$	Assumption
(1)	$\Theta' \;\; \mathsf{ctx}$	Inversion (Cwfuvar)
(2)	$\Theta \longrightarrow \Omega$	Assumption
(3)	$\Omega \;\; \mathsf{ctx}$	Assumption
(4)	$\Theta \Longrightarrow \Theta'$	Subderivation
	$(\Theta', \hat{\alpha}) \upharpoonright \Theta \longrightarrow \Omega$	Assumption
(5)	$\Theta' \upharpoonright \Theta \longrightarrow \Omega$	Inversion ($\upharpoonright$guessnotin)
	$\Omega' \;\; \mathsf{ctx}$	By i.h., using (1–5) and for some complete context Ω'
	$\Theta' \longrightarrow \Omega'$	"
	$\Omega \Longrightarrow \Omega'$	"
☞	$(\Omega', \hat{\alpha} = {\downarrow}\forall\alpha.\,{\uparrow}\alpha) \;\; \mathsf{ctx}$	By Cwfsolvedguess
☞	$\Theta', \hat{\alpha} \longrightarrow (\Omega', \hat{\alpha} = {\downarrow}\forall\alpha.\,{\uparrow}\alpha)$	By Csolveguess

☞ $\Omega \Longrightarrow (\Omega', \hat{\alpha} = \downarrow\forall\alpha. \uparrow\alpha)$ By Wcnewunsolvedguess

- **Case** $$\dfrac{\Theta \Longrightarrow \Theta'}{\Theta \Longrightarrow \Theta', \hat{\alpha} = P}\ \text{Wcnewsolvedguess}$$

  We add the $\hat{\alpha} = P$ context item onto the new context from the induction hypothesis.

	$\Theta', \hat{\alpha} = P$ ctx	Assumption
(1)	Θ' ctx	Inversion (Cwfunsolvedguess)
	$\Theta' \vdash P$ type$^+$	″
	P ground	″
(2)	$\Theta \longrightarrow \Omega$	Assumption
(3)	Ω ctx	Assumption
(4)	$\Theta \Longrightarrow \Theta'$	Subderivation
	$(\Theta', \hat{\alpha} = P) \upharpoonright \Theta \longrightarrow \Omega$	Assumption
(5)	$\Theta' \upharpoonright \Theta \longrightarrow \Omega$	Inversion ($\upharpoonright$guessnotin)
	Ω' ctx	By i.h., using (1–5) and for some complete context Ω'
	$\Theta' \longrightarrow \Omega'$	″
	$\Omega \Longrightarrow \Omega'$	″
	$\Omega' \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
☞	$(\Omega', \hat{\alpha} = P)$ ctx	By Cwfsolvedguess
☞	$\Theta', \hat{\alpha} \longrightarrow (\Omega', \hat{\alpha} = P)$	By Csolveguess
☞	$\Omega \Longrightarrow (\Omega', \hat{\alpha} = P)$	By Wcnewunsolvedguess

□

Lemma N.2 (Identical restricted contexts). *If* Θ' ctx *and* $\Theta \longrightarrow \Theta'$*, then* $\Theta'' \upharpoonright \Theta = \Theta'' \upharpoonright \Theta'$.

Proof. By rule induction on the $\Theta'' \upharpoonright \Theta$ judgment.

- **Case** $$\dfrac{}{\cdot \upharpoonright \cdot = \cdot}\ \upharpoonright\text{empty}$$

	$\cdot \longrightarrow \Theta'$	Assumption
	$\Theta' = \cdot$	Inversion (Cempty)
☞	$\cdot \upharpoonright \cdot = \cdot$	By $\upharpoonright$empty

- **Case** $$\dfrac{\Theta'' \upharpoonright \Theta = \Theta'''}{\Theta'', \alpha \upharpoonright \Theta, \alpha = \Theta''', \alpha}\ \upharpoonright\text{uvar}$$

$\alpha \in \Theta'$	By Lemma K.4 (Context extension maintains variables)
$\Theta'' \upharpoonright \Theta' = \Theta'''$	By i.h.

☞ $\Theta'', \alpha \upharpoonright \Theta' = \Theta''', \alpha$ By $\upharpoonright$uvar
(the α context item must appear last in Θ' by well-formedness of Θ')

- **Case** $$\dfrac{\Theta'' \upharpoonright \Theta = \Theta'''}{\Theta'', \hat{\alpha} [= P] \upharpoonright \Theta, \hat{\alpha} [= Q] = \Theta''', \hat{\alpha} [= P]} \upharpoonright\text{guessin}$$

$\hat{\alpha} [= R] \in \Theta'$ By Lemma K.4 (Context extension maintains variables)
$\Theta'' \upharpoonright \Theta' = \Theta'''$ By i.h.
☞ $\Theta'', \hat{\alpha} [= P] \upharpoonright \Theta' = \Theta''', \hat{\alpha} [= P]$ By $\upharpoonright$guessin
(the $\hat{\alpha} [= R]$ context item must appear in Θ' last by well-formedness of Θ')

- **Case** $$\dfrac{\Theta'' \upharpoonright \Theta = \Theta''' \qquad \hat{\alpha} [= Q] \notin \Theta}{\Theta'', \hat{\alpha} [= P] \upharpoonright \Theta = \Theta'''} \upharpoonright\text{guessnotin}$$

$\hat{\alpha} [= R] \notin \Theta'$ By Lemma K.4 (Context extension maintains variables)
$\Theta'' \upharpoonright \Theta' = \Theta'''$ By i.h.
☞ $\Theta'', \hat{\alpha} [= P] \upharpoonright \Theta' = \Theta'''$ By $\upharpoonright$guessnotin

□

N'.2 Statement

Theorem N.3 (Soundness of algorithmic typing). *If* Θ ctx*,* $\Theta \vdash \Gamma$ env*,* $\Theta' \longrightarrow \Omega$*, and* Ω ctx*, then:*

- *If* $\Theta; \Gamma \vdash v : P \dashv \Theta'$*, then* $\|\Theta\| ; \Gamma \vdash v : [\Omega]P$.
- *If* $\Theta; \Gamma \vdash t : N \dashv \Theta'$*, then* $\|\Theta\| ; \Gamma \vdash t : [\Omega]N$.
- *If* $\Theta; \Gamma \vdash s : N \gg M \dashv \Theta'$*,* $\Theta \vdash N\, \text{type}^-$*, and* $[\Theta]N = N$*, then* $\exists M'$ *such that* $\|\Theta\| \vdash [\Omega]M \cong^- M'$ *and* $\|\Theta\| ; \Gamma \vdash s : [\Omega]N \gg M'$.

Proof. By mutual induction with Theorem O.4 (Completeness of algorithmic typing), using the judgment ordering from Lemma J.1 (Isomorphic environments type the same terms).

- **Case** $$\dfrac{x : P \in \Gamma}{\Theta; \Gamma \vdash x : P \dashv \Theta} \text{Avar}$$

$x : P \in \Gamma$ Premise
P ground Typing environment only contains ground types
$x : [\Omega]P \in \Gamma$ By Lemma D.5 (Applying a context to a ground type)
☞ $\|\Theta\| ; \Gamma \vdash x : [\Omega]P$ By Dvar

- **Case** $$\frac{\Theta; \Gamma, x : P \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N \dashv \Theta'}\ \mathsf{A\lambda abs}$$

Θ ctx	Assumption
$\Theta \vdash \Gamma$ env	Assumption
$\Theta; \Gamma \vdash \lambda x : P.\, t : P \to N \dashv \Theta'$	Assumption
$\Theta \vdash P \to N$ type$^-$	By Lemma K.5 (Algorithmic typing is w.f)
$P \to N$ ground	"
$\Theta \vdash P$ type$^+$	Inversion (Twfarrow)
$P \to N$ ground	Assumption
P ground	By definition of ground
$\Theta \vdash \Gamma, x : P$ env	By Ewfvar
$\lVert\Theta\rVert ; \Gamma, x : P \vdash t : [\Omega]N$	By i.h. (term size decreases)
$\lVert\Theta\rVert ; \Gamma, x : P \vdash t : [\Omega]N$	By definition of $[-]-$
$\lVert\Theta\rVert ; \Gamma, x : [\Omega]P \vdash t : [\Omega]N$	By Lemma D.5 (Applying a context to a ground type)
$\lVert\Theta\rVert ; \Gamma \vdash \lambda x.t : [\Omega]P \to [\Omega]N$	By Dλabs
☞ $\lVert\Theta\rVert ; \Gamma \vdash \lambda x.t : [\Omega](P \to N)$	By definition of $[-]-$

- **Case** $$\frac{\Theta, \alpha; \Gamma \vdash t : N \dashv \Theta', \alpha}{\Theta; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N \dashv \Theta'}\ \mathsf{Agen}$$

Θ ctx	Assumption
Θ, α ctx	By Cwfuvar
$\Theta \vdash \Gamma$ env	Assumption
$\Theta, \alpha \vdash \Gamma$ env	Weakening
$\Theta' \longrightarrow \Omega$	Assumption
$\Theta', \alpha \longrightarrow \Omega, \alpha$	By Cuvar
Ω ctx	Assumption
Ω, α ctx	By Cwfuvar
$\lVert\Theta, \alpha\rVert ; \Gamma \vdash t : [\Omega, \alpha]N$	By i.h. (term size decreases)
$\lVert\Theta\rVert , \alpha; \Gamma \vdash t : [\Omega]N$	By definitions of $\lVert-\rVert$ and $[-]-$
$\lVert\Theta\rVert ; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, ([\Omega]N)$	By Dgen
☞ $\lVert\Theta\rVert ; \Gamma \vdash \Lambda\alpha.\, t : [\Omega](\forall\alpha.\, N)$	By definition of $[-]-$

- **Case** $$\frac{\Theta; \Gamma \vdash t : N \dashv \Theta'}{\Theta; \Gamma \vdash \{t\} : {\downarrow}N \dashv \Theta'}\ \mathsf{Athunk}$$

$\Theta; \Gamma \vdash t : N \dashv \Theta'$	Subderivation
$\lVert\Theta\rVert ; \Gamma \vdash t : [\Omega]N$	By i.h. (term size decreases)
$\lVert\Theta\rVert ; \Gamma \vdash \{t\} : {\downarrow}[\Omega]N$	By Dthunk
☞ $\lVert\Theta\rVert ; \Gamma \vdash t : [\Omega]{\downarrow}N$	By definition of $[-]-$

- **Case** $$\dfrac{\Theta; \Gamma \vdash v : P \dashv \Theta'}{\Theta; \Gamma \vdash \mathsf{return}\ v : \uparrow P \dashv \Theta'}\ \textsf{Areturn}$$

  Symmetric to Athunk case.

- **Case** $$\dfrac{\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta' \quad \Theta'; \Gamma \vdash s : M \gg \uparrow Q \dashv \Theta'' \quad \Theta'' \vdash P \leq^+ Q \dashv \Theta''' \\ \Theta''' \vdash [\Theta''']Q \leq^+ P \dashv \Theta^{(4)} \quad \Theta^{(5)} = \Theta^{(4)} \upharpoonright \Theta \quad \Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta^{(6)}}{\Theta; \Gamma \vdash \mathsf{let}\ x : P = v(s); t : N \dashv \Theta^{(6)}}\ \textsf{Aambiguouslet}$$

Use well-formedness of the first premise:

	Θ ctx	Assumption
	$\Theta \vdash \Gamma$ env	Assumption
	$\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta'$	Subderivation
	$\Theta \longrightarrow \Theta'$	By Lemma K.5 (Algorithmic typing is w.f)
	Θ' ctx	"
	$\Theta' \vdash \downarrow M$ type$^+$	"
	$\downarrow M$ ground	"

Now use the well-formedness of $\Theta'; \Gamma \vdash s : M \gg \uparrow Q \dashv \Theta''$:

	Θ' ctx	Above
(1)	$\Theta \Longrightarrow \Theta'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
	$\Theta' \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta' \vdash M$ type$^-$	Inversion (Twfshift$\downarrow$)
	M ground	By definition of ground
	$[\Theta']M = M$	By Lemma D.5 (Applying a context to a ground type)
(2)	$\Theta' \Longrightarrow \Theta''$	By Lemma K.5 (Algorithmic typing is w.f)
	Θ'' ctx	"
	$\Theta'' \vdash \uparrow Q$ type$^-$	"
	$[\Theta'']\uparrow Q = \uparrow Q$	"

Next use the well-formedness of $\Theta'' \vdash P \leq^+ Q \dashv \Theta'''$:

	Θ'' ctx	Above
	P ground	P annotation
	$[\Theta'']Q = Q$	By definition of $[-]-$
	Θ''' ctx	By Lemma K.5 (Algorithmic typing is w.f)
	$\Theta'' \longrightarrow \Theta'''$	"
	$[\Theta''']Q$ ground	"

And the well-formedness of $\Theta''' \vdash [\Theta''']Q \leq^+ P \dashv \Theta^{(4)}$:

	Θ''' ctx	Above
	$[\Theta''']Q$ ground	Above
	$[\Theta''']P = P$	By Lemma D.5 (Applying a context to a ground type)

$$\Theta^{(4)} \text{ ctx} \qquad \text{By Lemma K.5 (Algorithmic typing is w.f)}$$
$$\Theta''' \longrightarrow \Theta^{(4)} \qquad ''$$

Use the well-formedness of the restricted context:

$$\Theta \text{ ctx} \qquad \text{Above}$$
$$\Theta^{(4)} \text{ ctx} \qquad \text{Above}$$
$$\Theta'' \Longrightarrow \Theta''' \qquad \text{By Lemma C.1 } (\Longrightarrow \text{ subsumes } \longrightarrow)$$
$$\Theta''' \Longrightarrow \Theta^{(4)} \qquad \text{By Lemma C.1 } (\Longrightarrow \text{ subsumes } \longrightarrow)$$
$$\Theta \Longrightarrow \Theta^{(4)} \qquad \text{Applying Lemma C.4 (Weak context extension is transitive) to (1), (2), and above}$$
$$\Theta^{(5)} = \Theta^{(4)} \upharpoonright \Theta \qquad \text{Premise}$$

$$\Theta^{(5)} \text{ ctx} \qquad \text{By Lemma K.1 (Well-formedness of restricted contexts)}$$
(3) $$\Theta \longrightarrow \Theta^{(5)} \qquad ''$$
(4) $$\Theta^{(5)} \Longrightarrow \Theta^{(4)} \qquad ''$$

Finally use the well-formedness of $\Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta''''$:

$$\Theta \vdash \Gamma \text{ env} \qquad \text{Assumption}$$
$$\Theta \Longrightarrow \Theta^{(5)} \qquad \text{By Lemma C.1 } (\Longrightarrow \text{ subsumes } \longrightarrow)$$
$$\Theta^{(5)} \vdash \Gamma \text{ env} \qquad \text{By Lemma C.6 (Weak context extension preserves w.f. envs)}$$
$$\Theta \vdash P \text{ type}^{+} \qquad P \text{ annotation}$$
$$P \text{ ground} \qquad ''$$

$$\Theta^{(5)} \text{ ctx} \qquad \text{Above}$$
$$\Theta^{(5)} \vdash \Gamma, x : P \text{ env} \qquad \text{By Ewfvar}$$
$$\Theta^{(5)}; \Gamma; x : P \vdash t : N \dashv \Theta^{(6)} \qquad \text{Subderivation}$$

$$\Theta^{(6)} \text{ ctx} \qquad \text{By Lemma K.5 (Algorithmic typing is w.f)}$$
$$\Theta^{(5)} \longrightarrow \Theta^{(6)} \qquad ''$$

Use Lemma N.1 (Extended complete context) to obtain a complete context for the second to fourth judgments:

$$\Theta^{(5)} \longrightarrow \Theta^{(6)} \qquad \text{Above}$$
$$\Theta^{(6)} \longrightarrow \Omega \qquad \text{Assumption}$$
$$\Theta^{(5)} \longrightarrow \Omega \qquad \text{By Lemma D.3 (Context extension is transitive)}$$

$$\Theta^{(4)} \text{ ctx} \qquad \text{Above}$$
$$\Omega \text{ ctx} \qquad \text{Assumption}$$
$$\Theta^{(5)} \longrightarrow \Omega \qquad \text{Above}$$
$$\Theta^{(5)} \Longrightarrow \Theta^{(4)} \qquad \text{Above}$$
$$\Theta^{(5)} = \Theta^{(4)} \upharpoonright \Theta \qquad \text{Above}$$
$$\Theta^{(4)} \upharpoonright \Theta \longrightarrow \Omega \qquad \text{Substituting using above}$$
$$\Theta^{(4)} \upharpoonright \Theta^{(5)} \longrightarrow \Omega \qquad \text{By Lemma N.2 (Identical restricted contexts)}$$

$$\Omega' \text{ ctx} \qquad \text{By Lemma N.1 (Extended complete context)}$$
$$\Theta^{(4)} \longrightarrow \Omega' \qquad ''$$
$$\Omega \Longrightarrow \Omega' \qquad ''$$

$\Theta''' \longrightarrow \Theta^{(4)}$	Above
$\Theta''' \longrightarrow \Omega'$	By Lemma D.3 (Context extension is transitive)
$\Theta'' \longrightarrow \Theta'''$	Above
$\Theta'' \longrightarrow \Omega'$	By Lemma D.3 (Context extension is transitive)

Restrict Ω' such that Θ' extends to it:

$\Theta'' \Longrightarrow \Omega'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
$\Theta' \Longrightarrow \Omega'$	By Lemma C.4 (Weak context extension is transitive)
Let $\Omega'' = \Omega' \upharpoonright \Theta'$.	
Ω'' ctx	By Lemma K.1 (Well-formedness of restricted contexts)
$\Theta' \longrightarrow \Omega''$	"
$\Omega'' \Longrightarrow \Omega'$	"

Apply the induction hypothesis to the first premise:

Θ ctx	Above
$\Theta \vdash \Gamma$ env	Above
$\Theta' \longrightarrow \Omega''$	Above
Ω'' ctx	Above
$\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta'$	Subderivation
$\lVert\Theta\rVert ; \Gamma \vdash v : [\Omega'']{\downarrow}M$	By i.h. (term size decreases)
$\Omega'' \vdash {\downarrow}M$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)
$\Omega'' \Longrightarrow \Omega'$	Above
$[\Omega']M$ ground	By Lemma D.5 (Applying a context to a ground type)
Ω'' ctx	Above
Ω' ctx	Above
$\lVert\Omega''\rVert \vdash [\Omega'']M \cong^- [\Omega']M$	By Lemma F.3 ($\Longrightarrow$ leads to isomorphic types (ground))
$\lVert\Theta\rVert \vdash [\Omega'']M \cong^- [\Omega']M$	By Lemma O.1 (Weak context extension maintains variables) and Lemma K.4 (Context extension maintains variables)

Next apply the induction hypothesis to the spine premise:

Θ' ctx	Above
$\Theta' \vdash \Gamma$ env	Above
$\Theta'' \longrightarrow \Omega'$	Above
Ω' ctx	Above
$\Theta'; \Gamma \vdash s : M \gg {\uparrow}Q \dashv \Theta''$	Subderivation
$\Theta' \vdash M$ type$^-$	Above
$[\Theta']M = M$	Above
$\lVert\Theta\rVert ; \Gamma \vdash s : [\Omega']M \gg M'$	By i.h. (term size decreases)
$\lVert\Theta\rVert \vdash [\Omega']{\uparrow}Q \cong^- M'$	"
$M' = {\uparrow}Q'$	Declarative typing rules preserve shift structure
$\lVert\Theta\rVert ; \Gamma \vdash s : [\Omega']M \gg {\uparrow}Q'$	Substituting above
$\lVert\Theta\rVert ; \Gamma \vdash s : [\Omega'']M \gg {\uparrow}Q''$	By Lemma J.1 (Isomorphic environments type the same terms), using the fact that declarative typing rules preserve shift structure

$\|\Theta\| \vdash \uparrow Q' \cong^- \uparrow Q''$ — "

Show the third premise of Dambiguouslet, first by establishing one direction of the isomorphism:

(5)	Θ'' ctx	Shown above
(6)	$\Theta''' \longrightarrow \Omega'$	Above
(7)	$\Theta'' \vdash P$ type$^+$	Above
(8)	P ground	Above
	$\Theta'' \vdash \uparrow Q$ type$^-$	Above
(9)	$\Theta'' \vdash Q$ type$^+$	By inversion of Twfshift$\uparrow$
	$[\Theta'']\uparrow Q = \uparrow Q$	By Lemma K.5 (Algorithmic typing is w.f)
(10)	$[\Theta'']Q = Q$	By definition of $[-]-$
	$\|\Theta''\| \vdash P \leq^+ [\Omega']Q$	By (5 – 10) & Theorem F.6 (Soundness of algorithmic subtyping)

Now establish the other direction:

Θ''' ctx	Shown above
$\Theta^{(4)} \longrightarrow \Omega'$	Above
$\Theta''' \vdash [\Theta''']Q$ type$^+$	By Lemma E.1 (Applying context to the type preserves w.f.)
$[\Theta''']Q$ ground	Above
$\Theta''' \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
$[\Theta''']P = P$	By Lemma D.5 (Applying a context to a ground type)
$\|\Theta'''\| \vdash [\Theta''']Q \leq^+ [\Omega']P$	By Theorem F.6 (Soundness of algorithmic subtyping)

Show the fourth premise of Dambiguouslet:

$\|\Theta'''\| \vdash [\Theta''']Q \leq^+ P$	By Lemma D.5 (Applying a context to a ground type)
$\|\Theta'''\| \vdash [\Theta''']Q \cong^+ [\Omega']Q$	By Lemma F.5 ($\longrightarrow$ leads to isomorphic types (ground))
$\|\Theta'''\| \vdash [\Omega']Q \leq^+ P$	By Lemma B.7 (Declarative subtyping is transitive)
$\|\Theta''\| \vdash [\Omega']Q \leq^+ P$	By Lemma D.2 (Equality of declarative contexts)
$\|\Theta\| \vdash [\Omega']Q \leq^+ P$	By Lemma C.3 (Equality of declarative contexts (weak))
$\|\Theta\| \vdash P \leq^+ [\Omega']Q$	By Lemma C.3 (Equality of declarative contexts (weak))
$\|\Theta\| \vdash P \cong^+ [\Omega']Q$	We have shown the subtyping in both directions
$\|\Theta\| \vdash [\Omega']\uparrow Q \cong^- M'$	Above
$\|\Theta\| \vdash [\Omega']\uparrow Q \cong^- \uparrow Q'$	Substituting definition of Q' into above
$\|\Theta\| \vdash [\Omega']Q \cong^+ Q'$	Inversion ($\leq^{\pm}$Dshift$\uparrow$)
$\|\Theta\| \vdash P \cong^+ Q'$	By Lemma B.7 (Declarative subtyping is transitive)
$\|\Theta\| \vdash \uparrow P \leq^- \uparrow Q'$	By $\leq^{\pm}$Dshift$\uparrow$
$\|\Theta\| \vdash \uparrow P \leq^- \uparrow Q''$	By Lemma B.7 (Declarative subtyping is transitive)

And now show the final premise of Dambiguouslet:

	$\Theta \vdash N$ type$^-$	Assumption
(11)	$\Theta^{(5)}$ ctx	Above
	$\Theta \vdash \Gamma$ env	Assumption
	$\Theta^{(5)} \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta^{(5)} \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
	P ground	Above

(12)	$\Theta^{(5)} \vdash \Gamma, x : P\ \mathsf{env}$	By Ewfvar
(13)	$\Theta^{(6)} \longrightarrow \Omega$	Assumption
	$\|\Theta\| ; \Gamma, x : P \vdash t : [\Omega]N$	By (11–13) & i.h.

Finally apply Dambiguouslet:

☞	$\|\Theta\| ; \Gamma \vdash \mathsf{let}\ x : P = v(s); t : [\Omega]N$	By Dambiguouslet

- **Case**

$$\dfrac{\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta' \quad \Theta'; \Gamma \vdash s : M \gg \uparrow Q \dashv \Theta'' \qquad \mathrm{FEV}(Q) = \emptyset \qquad \Theta''' = \Theta'' \upharpoonright \Theta \qquad \Theta'''; \Gamma, x : Q \vdash t : N \dashv \Theta^{(4)}}{\Theta; \Gamma \vdash \mathsf{let}\ x = v(s); t : N \dashv \Theta^{(4)}}\ \mathsf{Aunambiguouslet}$$

As with the Aambiguouslet case, first use the well-formedness of the first premise:

$\Theta\ \mathsf{ctx}$	Assumption
$\Theta \vdash \Gamma\ \mathsf{env}$	Assumption
$\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta'$	Subderivation
$\Theta \longrightarrow \Theta'$	By Lemma K.5 (Algorithmic typing is w.f)
$\Theta'\ \mathsf{ctx}$	″
$\Theta' \vdash \downarrow M\ \mathsf{type}^+$	″
$\downarrow M\ \mathsf{ground}$	″

Now use the well-formedness of $\Theta'; \Gamma \vdash s : M \gg \uparrow Q \dashv \Theta''$:

	$\Theta'\ \mathsf{ctx}$	Above
(1)	$\Theta \Longrightarrow \Theta'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
	$\Theta' \vdash \Gamma\ \mathsf{env}$	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta' \vdash M\ \mathsf{type}^-$	Inversion (Twfshift↓)
	$M\ \mathsf{ground}$	By definition of ground
	$[\Theta']M = M$	By Lemma D.5 (Applying a context to a ground type)
(2)	$\Theta' \Longrightarrow \Theta''$	By Lemma K.5 (Algorithmic typing is w.f)
	$\Theta''\ \mathsf{ctx}$	″
	$\Theta'' \vdash \uparrow Q\ \mathsf{type}^-$	″

Use the well-formedness of the restricted context:

	$\Theta\ \mathsf{ctx}$	Above
	$\Theta''\ \mathsf{ctx}$	Above
	$\Theta \Longrightarrow \Theta''$	Applying Lemma C.4 (Weak context extension is transitive) to (1) and (2)
	$\Theta''' = \Theta'' \upharpoonright \Theta$	Premise
(3)	$\Theta'''\ \mathsf{ctx}$	By Lemma K.1 (Well-formedness of restricted contexts)
	$\Theta \longrightarrow \Theta'''$	″
	$\Theta''' \Longrightarrow \Theta''$	″

Finally use the well-formedness of $\Theta'''; \Gamma, x : P \vdash t : N \dashv \Theta^{(4)}$:

$\Theta \vdash \Gamma$ env	Assumption
$\Theta \Longrightarrow \Theta'''$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
$\Theta''' \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta \vdash P$ type$^+$	P annotation
P ground	″
Θ''' ctx	Above
$\Theta''' \vdash \Gamma, x : P$ env	By Ewfvar
$\Theta'''; \Gamma; x : P \vdash t : N \dashv \Theta^{(4)}$	Subderivation
$\Theta^{(4)}$ ctx	By Lemma K.5 (Algorithmic typing is w.f)
$\Theta''' \longrightarrow \Theta^{(4)}$	″

Use Lemma N.1 (Extended complete context) to obtain a complete context for the second judgment:

$\Theta''' \longrightarrow \Theta^{(4)}$	Above
$\Theta^{(4)} \longrightarrow \Omega$	Assumption
$\Theta''' \longrightarrow \Omega$	By Lemma D.3 (Context extension is transitive)
Θ'' ctx	Above
Ω ctx	Assumption
$\Theta''' \longrightarrow \Omega$	Above
$\Theta''' \Longrightarrow \Theta''$	Above
$\Theta''' = \Theta'' \upharpoonright \Theta$	Above
$\Theta'' \upharpoonright \Theta \longrightarrow \Omega$	Substituting using above
$\Theta'' \upharpoonright \Theta''' \longrightarrow \Omega$	By Lemma N.2 (Identical restricted contexts)
Ω' ctx	By Lemma N.1 (Extended complete context)
$\Theta'' \longrightarrow \Omega'$	″
$\Omega \Longrightarrow \Omega'$	

Restrict Ω' such that Θ' extends to it:

$\Theta'' \Longrightarrow \Omega'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
$\Theta' \Longrightarrow \Omega'$	By Lemma C.4 (Weak context extension is transitive)
Let $\Omega'' = \Omega' \upharpoonright \Theta'$.	
Ω'' ctx	By Lemma K.1 (Well-formedness of restricted contexts)
$\Theta' \longrightarrow \Omega''$	″
$\Omega'' \Longrightarrow \Omega'$	″

Apply the induction hypothesis to the first premise:

Θ ctx	Above
$\Theta \vdash \Gamma$ env	Above
$\Theta' \longrightarrow \Omega''$	Above
Ω'' ctx	Above
$\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta'$	Subderivation
$\lVert\Theta\rVert ; \Gamma \vdash v : [\Omega'']\downarrow M$	By i.h. (term size decreases)

	$\Omega'' \vdash \downarrow M \ \text{type}^-$	By Lemma E.1 (Applying context to the type preserves w.f.)
	$\Omega'' \Longrightarrow \Omega'$	Above
	$[\Omega']M \ \text{ground}$	By Lemma D.5 (Applying a context to a ground type)
	$\Omega'' \ \text{ctx}$	Above
	$\Omega' \ \text{ctx}$	Above
	$\|\Omega''\| \vdash [\Omega'']M \cong^- [\Omega']M$	By Lemma F.3 ($\Longrightarrow$ leads to isomorphic types (ground))
	$\|\Theta\| \vdash [\Omega'']M \cong^- [\Omega']M$	By Lemma O.1 (Weak context extension maintains variables) and Lemma K.4 (Context extension maintains variables)

Next apply the induction hypothesis to the spine premise:

	$\Theta' \ \text{ctx}$	Above
	$\Theta' \vdash \Gamma \ \text{env}$	Above
	$\Theta'' \longrightarrow \Omega'$	Above
	$\Omega' \ \text{ctx}$	Above
	$\Theta'; \Gamma \vdash s : M \gg \uparrow Q \dashv \Theta''$	Subderivation
	$\Theta' \vdash M \ \text{type}^-$	Above
	$[\Theta']M = M$	Above
	$\|\Theta\| ; \Gamma \vdash s : [\Omega']M \gg M'$	By i.h. (term size decreases)
	$\|\Theta\| \vdash [\Omega']\uparrow Q \cong^- M'$	$''$
	$M' = \uparrow Q'$	Declarative typing rules preserve shift structure
	$\|\Theta\| ; \Gamma \vdash s : [\Omega']M \gg \uparrow Q'$	Substituting above equation
	$\|\Theta\| ; \Gamma \vdash s : [\Omega'']M \gg \uparrow Q''$	By Lemma J.1 (Isomorphic environments type the same terms), using the fact that declarative typing rules preserve shift structure
	$\|\Theta\| \vdash \uparrow Q' \cong^- \uparrow Q''$	$''$

Next apply the induction hypothesis to the last premise:

	$\Theta \vdash N \ \text{type}^-$	Assumption
(4)	$\Theta''' \ \text{ctx}$	Above
	$\Theta \vdash \Gamma \ \text{env}$	Assumption
	$\Theta''' \vdash \Gamma \ \text{env}$	By Lemma C.6 (Weak context extension preserves w.f. envs)
	$\Theta''' \vdash Q \ \text{type}^+$	By Lemma D.4 (Context extension preserves w.f.)
	$\text{FEV}(Q) = \emptyset$	Premise
	$Q \ \text{ground}$	By definition of ground
(5)	$\Theta''' \vdash \Gamma, x : Q \ \text{env}$	By Ewfvar
	$\Theta^{(4)} \longrightarrow \Omega$	Assumption
	$\Theta''' \longrightarrow \Theta^{(4)}$	Above
(6)	$\Theta''' \longrightarrow \Omega$	By Lemma D.3 (Context extension is transitive)
	$\|\Theta\| ; \Gamma, x : Q \vdash t : [\Omega]N$	By (4–6) & i.h.

Rework the declarative judgment we got from the induction hypothesis to match the form we need to apply Dunambiguouslet:

	$\|\Theta\| \vdash [\Omega']\uparrow Q \cong^- M'$	Above
	$\|\Theta\| \vdash \uparrow Q \cong^- M'$	By Lemma D.5 (Applying a context to a ground type)
	$\|\Theta\| \vdash \uparrow Q \cong^- \uparrow Q'$	Substituting in the definition of Q'
	$\|\Theta\| \vdash \uparrow Q \cong^- \uparrow Q''$	By Lemma B.7 (Declarative subtyping is transitive)

$$\|\Theta\| \vdash Q \cong^+ Q''$$ Inversion ($\leq^\pm$Ashift$\uparrow$)

$$\|\Theta\|\,; \Gamma, x : Q'' \vdash t : [\Omega]N$$ Using Lemma J.1 (Isomorphic environments type the same terms) to change the typing environment

Now show that for all positive types P, if $\|\Theta\|\,; \Gamma \vdash s : [\Omega']M \gg \uparrow P$ then $\|\Theta\| \vdash Q'' \cong^+ P$. Let P be an arbitrary positive type such that $\|\Theta\|\,; \Gamma \vdash s : [\Omega']M \gg \uparrow P$.

	$\|\Theta'\|\,; \Gamma \vdash s : [\Omega']M \gg \uparrow P$	By Lemma D.2 (Equality of declarative contexts)
	$\Theta'; \Gamma \vdash s : M \gg \uparrow R \dashv \hat{\Theta}''$	By Theorem O.4 (Completeness of algorithmic typing), for some R and $\hat{\Theta}''$ (term size decreases)
(7)	$[\Omega']\uparrow R = \uparrow P$	"
	$\Theta'; \Gamma \vdash s : M \gg \uparrow Q \dashv \Theta''$	Subderivation
	$\uparrow R = \uparrow Q$	By Lemma H.1 (Algorithmic subtyping is deterministic)
	$[\Omega']\uparrow Q = [\Omega']\uparrow R$	Applying Ω' to both sides
	$[\Omega']\uparrow Q = \uparrow P$	Substituting using (7)
(9)	$[\Omega']Q = P$	By definition of $[-]-$
	$\|\Theta\| \vdash P \leq^+ P$	By Lemma B.1 (Declarative subtyping is reflexive)
	$\|\Theta\| \vdash P \cong^+ P$	By definition of $\cong^\pm$
	$\|\Theta\| \vdash [\Omega']Q \cong^+ P$	Substituting using (8)
	$\|\Theta\| \vdash [\Omega']Q \cong^+ Q'$	Above
	$\|\Theta\| \vdash Q'' \cong^+ P$	Applying Lemma B.7 (Declarative subtyping is transitive) twice

Finally apply Dunambiguouslet:

☞ $\|\Theta\|\,; \Gamma \vdash \mathsf{let}\ x = v(s); t : [\Omega]N$ By Dunambiguouslet

- **Case**

$$\frac{}{\Theta; \Gamma \vdash \epsilon : N \gg N \dashv \Theta}\ \mathsf{Aspinenil}$$

☞ $\|\Theta\|\,; \Gamma \vdash \epsilon : [\Omega]N \gg [\Omega]N$ By Dspinenil

☞ $\|\Theta\| \vdash [\Omega]N \cong^- [\Omega]N$ By Lemma B.1 (Declarative subtyping is reflexive)

- **Case**

$$\frac{\Theta; \Gamma \vdash v : P \dashv \Theta' \qquad \Theta' \vdash P \leq^+ [\Theta']Q \dashv \Theta'' \qquad \Theta''; \Gamma \vdash s : [\Theta'']N \gg M \dashv \Theta'''}{\Theta; \Gamma \vdash v, s : Q \to N \gg M \dashv \Theta'''}\ \mathsf{Aspinecons}$$

$\Theta \vdash Q \to N\ \mathsf{type}^-$	Assumption
$\Theta \vdash Q\ \mathsf{type}^+$	Inversion (Twfarrow)
$\Theta \vdash N\ \mathsf{type}^-$	Inversion (Twfarrow)
$[\Theta](Q \to N) = Q \to N$	Assumption
$[\Theta]Q \to [\Theta]N = Q \to N$	By definition of $[-]-$
$[\Theta]Q = Q$	By equality
$\Theta\ \mathsf{ctx}$	Assumption
$\Theta \vdash \Gamma\ \mathsf{env}$	Assumption
$\Omega\ \mathsf{ctx}$	Assumption

Apply typing well-formedness to the first premise:

$$\begin{array}{rll}
\Theta & \longrightarrow \Theta' & \text{By Lemma K.5 (Algorithmic typing is w.f)} \\
\Theta' & \text{ctx} & '' \\
\Theta' \vdash P & \text{type}^+ & '' \\
P & \text{ground} & ''
\end{array}$$

Now apply the well-formedness of subtyping to the second premise:

$$\begin{array}{rll}
\Theta' & \text{ctx} & \text{Above} \\
P & \text{ground} & \text{Above} \\
[\Theta'][\Theta']Q = & [\Theta']Q & \text{By Lemma D.6 (Context application is idempotent)} \\
\\
\Theta'' & \text{ctx} & \text{By Lemma E.2 (Algorithmic subtyping is w.f.)} \\
\Theta' & \longrightarrow \Theta'' & '' \\
[\Theta''][\Theta']Q & \text{ground} & '' \\
[\Theta'']Q & \text{ground} & \text{By Lemma D.8 (Extending context preserves groundness)}
\end{array}$$

Apply typing well-formedness to the last premise:

$$\begin{array}{rll}
\Theta'' \vdash N & \text{type}^- & \text{By Lemma D.4 (Context extension preserves w.f.)} \\
\Theta'' \vdash [\Theta'']N & \text{type}^- & \text{By Lemma E.1 (Applying context to the type preserves w.f.)} \\
[\Theta''][\Theta'']N = & [\Theta'']N & \text{By Lemma D.6 (Context application is idempotent)} \\
\Theta \vdash & \Gamma\ \text{env} & \text{Assumption} \\
\Theta''' \vdash & \Gamma\ \text{env} & \text{By Lemma C.6 (Weak context extension preserves w.f. envs)} \\
\\
\Theta'' & \Longrightarrow \Theta''' & \text{By Lemma K.5 (Algorithmic typing is w.f)} \\
\Theta''' & \text{ctx} & ''
\end{array}$$

Restrict Ω such that Θ'' extends to it:

$$\begin{array}{rll}
\Theta''' & \longrightarrow \Omega & \text{Assumption} \\
\Omega & \text{ctx} & \text{Assumption} \\
\Theta''' & \Longrightarrow \Omega & \text{By Lemma C.1 } (\Longrightarrow \text{ subsumes } \longrightarrow) \\
\Theta'' & \Longrightarrow \Omega & \text{By Lemma C.4 (Weak context extension is transitive)} \\
\\
\Omega \upharpoonright \Theta'' & \text{ctx} & \text{By Lemma K.1 (Well-formedness of restricted contexts)} \\
\Theta'' & \longrightarrow \Omega \upharpoonright \Theta'' & '' \\
\Omega \upharpoonright \Theta'' & \Longrightarrow \Omega & '' \\
\\
\Theta' & \Longrightarrow \Omega & \text{By Lemma C.1 } (\Longrightarrow \text{ subsumes } \longrightarrow) \\
& & \quad \text{and Lemma C.4 (Weak context extension is transitive)} \\
\Omega \vdash P & \text{type}^+ & \text{By Lemma C.5 (Weak context extension preserves well-formedness)} \\
\|(\Omega \upharpoonright \Theta'')\| \vdash [\Omega]P \cong^+ & [(\Omega \upharpoonright \Theta'')]P & \text{By Lemma F.3 } (\Longrightarrow \text{ leads to isomorphic types (ground))} \\
\Theta & \longrightarrow \Omega \upharpoonright \Theta'' & \text{By Lemma D.3 (Context extension is transitive)} \\
\|\Theta\| \vdash [\Omega]P \cong^+ & [(\Omega \upharpoonright \Theta'')]P & \text{By Lemma D.2 (Equality of declarative contexts)}
\end{array}$$

Applying the induction hypothesis to the first premise:

$$\begin{array}{ll}
\|\Theta\|\,;\Gamma \vdash v : [(\Omega \upharpoonright \Theta'')]P & \text{By i.h. (term size decreases)} \\
\|\Theta\|\,;\Gamma \vdash v : P & \text{By Lemma D.5 (Applying a context to a ground type)}
\end{array}$$

Applying soundness to the second premise:

$\Theta' \longrightarrow \Omega \upharpoonright \Theta''$	By transitivity
$\Theta' \vdash [\Theta']Q\ \ \text{type}^+$	By Lemma D.4 (Context extension preserves w.f.) and Lemma K.3 (Substitution preserves well-formedness of types)
$\|\Theta'\| \vdash P \leq^+ [\Omega \upharpoonright \Theta''][\Theta']Q$	By Theorem F.6 (Soundness of algorithmic subtyping)
$\|\Theta'\| \vdash P \leq^+ [\Omega \upharpoonright \Theta'']Q$	By Lemma F.4 ($\longrightarrow$ leads to isomorphic types)
$\|\Theta\| \vdash P \leq^+ [\Omega]Q$	By Lemma F.3 ($\Longrightarrow$ leads to isomorphic types (ground)) and Lemma D.2 (Equality of declarative contexts)

Apply the induction hypothesis to the last premise:

$\|\Theta''\|\,;\Gamma \vdash s : [\Omega][\Theta'']N \gg M'$	By i.h. (term size decreases)
$\|\Theta''\| \vdash [\Omega]M \cong^- M'$	$''$

Reworking the spine declarative judgment:

$\|\Theta''\| \vdash [\Omega][\Theta'']N \cong^- [\Omega]N$	By Lemma F.2 ($\Longrightarrow$ leads to isomorphic types)
$\|\Theta''\|\,;\Gamma \vdash s : [\Omega]N \gg M''$	By Lemma J.1 (Isomorphic environments type the same terms)
$\|\Theta''\| \vdash M' \cong^- M''$	$''$
$\|\Theta\|\,;\Gamma \vdash s : [\Omega]N \gg M''$	By Lemma D.2 (Equality of declarative contexts)

Applying the declarative judgment, we have:

	$\|\Theta\|\,;\Gamma \vdash v, s : [\Omega]Q \to [\Omega]N \gg [\Omega]M$	By Dspinecons
☞	$\|\Theta\|\,;\Gamma \vdash v, s : [\Omega](Q \to N) \gg M''$	By definition of $[-]-$
	$\|\Theta''\| \vdash [\Omega]M \cong^- M''$	By Lemma B.7 (Declarative subtyping is transitive)
☞	$\|\Theta\| \vdash [\Omega]M \cong^- M''$	By Lemma D.2 (Equality of declarative contexts)

- **Case** $\dfrac{\Theta; \Gamma \vdash s : N \gg M \dashv \Theta' \qquad \alpha \notin \mathrm{FUV}(N)}{\Theta; \Gamma \vdash s : (\forall \alpha.\, N) \gg M \dashv \Theta'}$ Aspinetypeabsnotin

	$\Theta \vdash \forall \alpha.\, N\ \ \text{type}^-$	Assumption
	$\Theta, \alpha \vdash N\ \ \text{type}^-$	Inversion (Twfforall)
(1)	$\Theta \vdash N\ \ \text{type}^-$	By Lemma K.2 (Type well-formed with type variable removed) and $\alpha \notin \mathrm{FUV}(N)$
	$[\Theta](\forall \alpha.\, N) = \forall \alpha.\, N$	Assumption
	$\forall \alpha.\, [\Theta]N = \forall \alpha.\, N$	By definition of $[-]-$
(2)	$[\Theta]N = N$	By equality
(3)	$\Theta\ \ \text{ctx}$	Assumption
(4)	$\Theta \vdash \Gamma\, \text{env}$	Assumption
(5)	$\Theta' \longrightarrow \Omega$	Assumption
(6)	$\Omega\ \ \text{ctx}$	Assumption

Apply the induction hypothesis:

	$\|\Theta\| ; \Gamma \vdash s : [\Omega]N \gg M'$	By i.h. (term size stays the same and the number of prenex quantifiers decreases)
☞	$\|\Theta\| \vdash [\Omega]M \cong^{-} M'$	"

Let P be an arbitrary positive type, such that $\Omega \vdash P\ \text{type}^{+}$:

$[P/\alpha]N = N$	As $\alpha \notin FUV(N)$
$\|\Theta\| ; \Gamma \vdash s : [\Omega][P/\alpha]N \gg M'$	By equality
$\|\Theta\| ; \Gamma \vdash s : [P/\alpha][\Omega]N \gg M'$	By definition of [–]–

Applying the declarative judgment:

	$\|\Theta\| ; \Gamma \vdash s : \forall\alpha.\, [\Omega]N \gg M'$	By Dspinetypeabs
☞	$\|\Theta\| ; \Gamma \vdash s : [\Omega](\forall\alpha.\, N) \gg M'$	By definition of [–]–

- **Case** $\dfrac{\Theta, \hat{\alpha}; \Gamma \vdash s : [\hat{\alpha}/\alpha]N \gg M \dashv \Theta', \hat{\alpha}\,[= P] \qquad \alpha \in FUV(N)}{\Theta; \Gamma \vdash s : (\forall\alpha.\, N) \gg M \dashv \Theta', \hat{\alpha}\,[= P]}$ Aspinetypeabsin

	$\Theta \vdash \forall\alpha.\, N\ \text{type}^{-}$	Assumption
	$\Theta, \alpha \vdash N\ \text{type}^{-}$	By Twfforall
(1)	$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N\ \text{type}^{-}$	By Lemma K.3 (Substitution preserves well-formedness of types)
	$[\Theta](\forall\alpha.\, N) = \forall\alpha.\, N$	Assumption
	$[\Theta]N = N$	By definition of [–]–
	$[\Theta][\hat{\alpha}/\alpha]N = [\hat{\alpha}/\alpha]N$	$\hat{\alpha}$ fresh
(2)	$[\Theta, \hat{\alpha}][\hat{\alpha}/\alpha]N = [\hat{\alpha}/\alpha]N$	By definition of [–]–
	$\Theta\ \text{ctx}$	Assumption
(3)	$\Theta, \hat{\alpha}\ \text{ctx}$	By Cwfunsolvedguess
	$\Theta \vdash \Gamma\ \text{env}$	Assumption
	$\Theta \Longrightarrow \Theta, \hat{\alpha}$	By Wcnewunsolvedguess
(4)	$\Theta, \hat{\alpha} \vdash \Gamma\ \text{env}$	By Lemma C.6 (Weak context extension preserves w.f. envs)
(5)	$\Omega\ \text{ctx}$	Assumption

Use well-formedness of the subderivation:

$\Theta, \hat{\alpha} \Longrightarrow \Theta, \hat{\alpha}\,[= P]$	By Lemma K.5 (Algorithmic typing is w.f)

Obtain the solution to $\hat{\alpha}$ from the complete context:

$\Theta', \hat{\alpha}\,[= P] \longrightarrow \Omega$	Assumption
$\Omega = \Omega', \hat{\alpha} = P'$	Inversion (Csolveguess), since Ω is a complete context
$\Omega', \hat{\alpha} = P'\ \text{ctx}$	Above
$\Omega' \vdash P'\ \text{type}^{+}$	Inversion (Cwfsolvedguess)
$P'\ \text{ground}$	"
$\Omega \vdash P'\ \text{type}^{+}$	By Lemma A.2 (Term well-formedness weakening)
$\Theta', \hat{\alpha}\,[= P] \longrightarrow \Omega$	Assumption
$\Theta', \hat{\alpha}\,[= P] \vdash P'\ \text{type}^{+}$	By Lemma K.4 (Context extension maintains variables)
$\Theta, \hat{\alpha} \vdash P'\ \text{type}^{+}$	By Lemma O.1 (Weak context extension maintains variables)
$\|\Theta\| \vdash P'\ \text{type}^{+}$	Since P' ground

Apply the induction hypothesis:

	$\|\Theta\| ; \Gamma \vdash s : [\Omega][\hat{\alpha}/\alpha]N \gg M'$	By i.h. (Term size stays the same and the number of prenex universal quantifiers decreases. The substitution replaces a positive type by another positive type, so cannot add or remove prenex universal quantifiers.)
☞	$\|\Theta\| \vdash [\Omega]M \cong^{-} M'$	"
	$\|\Theta\| ; \Gamma \vdash s : [\Omega', \hat{\alpha} = P'][\hat{\alpha}/\alpha]N \gg M'$	Substituting $\Omega = \Omega', \hat{\alpha} = P'$ from above
	$\|\Theta\| ; \Gamma \vdash s : [P'/\alpha][\Omega', \hat{\alpha} = P']N \gg M'$	By definition of $[-]-$
	$\|\Theta\| ; \Gamma \vdash s : [P'/\alpha][\Omega]N \gg M'$	Substituting $\Omega = \Omega', \hat{\alpha} = P'$ from above
	$\|\Theta\| ; \Gamma \vdash s : (\forall\alpha. [\Omega]N) \gg M'$	By Dspinetypeabs
☞	$\|\Theta\| ; \Gamma \vdash s : [\Omega](\forall\alpha. N) \gg M'$	By definition of substitution

□

O' Completeness of typing

O'.1 Lemmas

Lemma O.1 (Weak context extension maintains variables). *If* $\Theta \implies \Theta'$ *then* $\mathrm{FEV}(\Theta) \subseteq \mathrm{FEV}(\Theta')$ *and* $\mathrm{FUV}(\Theta) = \mathrm{FUV}(\Theta')$.

Proof. All rules ensure the left-hand side and right-hand side contexts have the same set of free universal variables. Wcempty, Wcuvar, Wcunsolvedguess, Wcsolveguess, and Wcsolvedguess ensure the left-hand side and right-hand side contexts have the same set of existential variables. The right-hand side context in the Wcnewunsolvedguess and Wcnewunsolvedguess rules have a set of existential variables that is a superset of the set of existential variables on the left-hand side context. □

Lemma O.2 (Reversing context extension from a complete context). *If* $\Omega \longrightarrow \Theta$ *then* $\Theta \longrightarrow \Omega$.

Proof. By rule induction on the $\Omega \longrightarrow \Theta$ judgment:

- **Case**

$$\frac{}{\cdot \longrightarrow \cdot}\ \mathsf{Cempty}$$

☞ $\cdot \longrightarrow \cdot$ Assumption

- **Case**

$$\frac{\Omega \longrightarrow \Theta}{\Omega, \alpha \longrightarrow \Theta, \alpha}\ \mathsf{Cuvar}$$

$\Omega \longrightarrow \Theta$	Subderivation
$\Theta \longrightarrow \Omega$	By i.h.

☞ $\Theta, \alpha \longrightarrow \Omega, \alpha$ By Cuvar

- **Case**

$$\frac{\Omega \longrightarrow \Theta}{\Omega, \hat{\alpha} \longrightarrow \Theta, \hat{\alpha}}\ \textsf{Cunsolvedguess}$$

Impossible since the LHS must be a complete context.

- **Case**

$$\frac{\Omega \longrightarrow \Theta}{\Omega, \hat{\alpha} \longrightarrow \Theta, \hat{\alpha} = P}\ \textsf{Csolveguess}$$

Impossible since the LHS must be a complete context.

- **Case**

$$\frac{\Omega \longrightarrow \Theta \qquad \|\Omega\| \vdash P \cong^{+} Q}{\Omega, \hat{\alpha} = P \longrightarrow \Theta, \hat{\alpha} = Q}\ \textsf{Csolvedguess}$$

	$\Omega \longrightarrow \Theta$	Subderivation
	$\Theta \longrightarrow \Omega$	By i.h.
	$\|\Omega\| \vdash P \cong^{+} Q$	Premise
	$\|\Theta\| \vdash P \cong^{+} Q$	By Lemma D.2 (Equality of declarative contexts)
	$\|\Theta\| \vdash Q \cong^{+} P$	By definition of $- \vdash - \cong^{\pm} -$
☞	$\Theta, \hat{\alpha} = Q \longrightarrow \Omega, \hat{\alpha} = P$	By Csolvedguess

□

Lemma O.3 (Pulling back restricted contexts). *If* $\Theta \longrightarrow \Theta'$ *and* $\Theta' \upharpoonright \Theta'' \longrightarrow \Theta'''$*, then* $\Theta \upharpoonright \Theta'' \longrightarrow \Theta'''$.

Proof. By rule induction on the $\Theta \longrightarrow \Theta'$ judgment:

- **Case**

$$\frac{}{\cdot \longrightarrow \cdot}\ \textsf{Cempty}$$

☞ $\cdot \upharpoonright \Theta'' \longrightarrow \Theta'''$ Assumption

- **Case**

$$\frac{\Theta \longrightarrow \Theta'}{\Theta, \alpha \longrightarrow \Theta', \alpha}\ \textsf{Cuvar}$$

$\Theta', \alpha \upharpoonright \Theta'' \longrightarrow \Theta'''$	Assumption
$\Theta' \upharpoonright \bar{\Theta}'' \longrightarrow \bar{\Theta}'''$	Inversion ($\upharpoonright$uvar)
$\Theta'' = \bar{\Theta}'', \alpha$	"
$\Theta''' = \bar{\Theta}''', \alpha$	"
$\Theta \longrightarrow \Theta'$	Subderivation
$\Theta \upharpoonright \bar{\Theta}'' \longrightarrow \bar{\Theta}'''$	By i.h.

$$
\begin{array}{rll}
 & (\Theta \upharpoonright \bar{\Theta}''), \alpha \longrightarrow \bar{\Theta}''', \alpha & \text{By Cuvar} \\
☞ & \Theta, \alpha \upharpoonright \Theta'' \longrightarrow \Theta''' & \text{By } \upharpoonright\text{uvar}
\end{array}
$$

- **Case** $\dfrac{\Theta \longrightarrow \Theta'}{\Theta, \hat{\alpha} \longrightarrow \Theta', \hat{\alpha}}$ Cunsolvedguess

- **Case** $\dfrac{\Theta \longrightarrow \Theta' \qquad \|\Theta\| \vdash P \cong^{+} Q}{\Theta, \hat{\alpha} = P \longrightarrow \Theta', \hat{\alpha} = Q}$ Csolvedguess

  Prove these two cases together.

$\Theta', \hat{\alpha}\,[= P] \upharpoonright \Theta'' \longrightarrow \Theta'''$ Assumption

  Taking cases on whether $\hat{\alpha}\,[= Q] \in \Theta''$:

  - **Case** $\hat{\alpha}\,[= Q] \in \Theta''$:

$$
\begin{array}{rll}
 & \Theta' \upharpoonright \bar{\Theta}'' \longrightarrow \bar{\Theta}''' & \text{Inversion } (\upharpoonright\text{guessin}) \\
 & \Theta'' = \bar{\Theta}'', \hat{\alpha}\,[= Q] & '' \\
 & \Theta''' = \bar{\Theta}''', \hat{\alpha}\,[= P] & '' \\
 & \Theta \upharpoonright \bar{\Theta}'' \longrightarrow \bar{\Theta}''' & \text{By i.h.} \\
 & (\Theta \upharpoonright \bar{\Theta}''), \hat{\alpha}\,[= P] \longrightarrow \bar{\Theta}''', \hat{\alpha}\,[= P] & \text{By Cunsolvedguess/ Csolvedguess} \\
☞ & \Theta, \hat{\alpha}\,[= P] \upharpoonright \Theta'' \longrightarrow \Theta''' & \text{By } \upharpoonright\text{guessin}
\end{array}
$$

  - **Case** $\hat{\alpha}\,[= Q] \notin \Theta''$:

$$
\begin{array}{rll}
 & \Theta' \upharpoonright \Theta'' \longrightarrow \Theta''' & \text{Inversion } (\upharpoonright\text{guessnotin}) \\
 & \Theta \upharpoonright \Theta'' \longrightarrow \Theta''' & \text{By i.h.} \\
☞ & \Theta, \hat{\alpha}\,[= P] \upharpoonright \Theta'' \longrightarrow \Theta''' & \text{By } \upharpoonright\text{guessnotin}
\end{array}
$$

□

O'.2 Statement

Theorem O.4 (Completeness of algorithmic typing). *If* Θ ctx, $\Theta \vdash \Gamma$ env, $\Theta \longrightarrow \Omega$, *and* Ω ctx, *then:*

- *If* $\|\Theta\|\,; \Gamma \vdash v : P$ *then* $\exists \Theta'$ *such that* $\Theta; \Gamma \vdash v : P \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.
- *If* $\|\Theta\|\,; \Gamma \vdash t : N$ *then* $\exists \Theta'$ *such that* $\Theta; \Gamma \vdash t : N \dashv \Theta'$ *and* $\Theta' \longrightarrow \Omega$.
- *If* $\|\Theta\|\,; \Gamma \vdash s : [\Omega]N \gg M$, $\Theta \vdash N\,\text{type}^{-}$, *and* $[\Theta]N = N$, *then* $\exists \Theta', \Omega'$ *and* M' *such that* $\Theta; \Gamma \vdash s : N \gg M' \dashv \Theta'$, $\Omega \Longrightarrow \Omega'$, $\Theta' \longrightarrow \Omega'$, $\|\Theta\| \vdash [\Omega']M' \cong^{-} M$, $[\Theta']M' = M'$, *and* Ω' ctx.

Proof. By mutual induction with Theorem N.3 (Soundness of algorithmic typing), using the same judgment ordering as in Lemma J.1 (Isomorphic environments type the same terms).

- **Case** $$\dfrac{x : P \in \Gamma}{\|\Theta\| \,; \Gamma \vdash x : P}\ \mathsf{Dvar}$$

	$x : P \in \Gamma$	Premise
☞	$\Theta; \Gamma \vdash x : P \dashv \Theta$	By Avar
☞	$\Theta \longrightarrow \Omega$	Assumption

- **Case** $$\dfrac{\|\Theta\| \,; \Gamma, x : P \vdash t : N}{\|\Theta\| \,; \Gamma \vdash \lambda x : P.\, t : P \rightarrow N}\ \mathsf{D\lambda abs}$$

	$\Theta \vdash P \rightarrow N$ type$^-$	Assumption
	$\Theta \vdash N$ type$^-$	Inversion (Twfarrow)
	$P \rightarrow N$ ground	Assumption
	N ground	By definition of ground
	Θ ctx	Assumption
	$\Theta \vdash \Gamma$ env	Assumption
	$\Theta \vdash P \rightarrow N$ type$^-$	Assumption
	$\Theta \vdash P$ type$^+$	Inversion (Twfarrow)
	$P \rightarrow N$ ground	Assumption
	P ground	By definition of ground
	$\Theta \vdash \Gamma, x : P$ env	By Ewfvar
	$\Theta \longrightarrow \Omega$	Assumption
	Ω ctx	Assumption
	$\Theta; \Gamma, x : P \vdash t : N \dashv \Theta'$	By i.h., for some context Θ' (term size decreases)
☞	$\Theta' \longrightarrow \Omega$	"
☞	$\Theta; \Gamma \vdash \lambda x : P.\, t : P \rightarrow N \dashv \Theta'$	By Aλabs

- **Case** $$\dfrac{\|\Theta\| \,, \alpha; \Gamma \vdash t : N}{\|\Theta\| \,; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N}\ \mathsf{Dgen}$$

$\|\Theta, \alpha\| \,; \Gamma \vdash t : N$	By definition of $\|-\|$
$\Theta \vdash \forall\alpha.\, N$ type$^-$	Assumption
$\Theta, \alpha \vdash N$ type$^-$	By Twfforall
$\forall\alpha.\, N$ ground	Assumption
N ground	By definition of ground

$$\begin{array}{rll}
& \Theta\ \mathsf{ctx} & \text{Assumption} \\
& \Theta, \alpha\ \mathsf{ctx} & \text{By Cwfuvar} \\
& \Theta \vdash \Gamma\,\mathsf{env} & \text{Assumption} \\
& \Theta \longrightarrow \Omega & \text{Assumption} \\
& \Theta, \alpha \longrightarrow \Omega, \alpha & \text{By Cuvar} \\
& \Omega\ \mathsf{ctx} & \text{Assumption} \\
& \Omega, \alpha\ \mathsf{ctx} & \text{By Cwfuvar} \\
& \Theta, \alpha; \Gamma \vdash t : N \dashv \Theta' & \text{By i.h., for some context } \Theta' \text{ (term size decreases)} \\
& \Theta' \longrightarrow \Omega, \alpha & '' \\
& \Theta' = \Theta'', \alpha & \text{Inversion (Cuvar)for some context } \Theta'' \\
☞ & \Theta'' \longrightarrow \Omega & '' \\
& \Theta, \alpha; \Gamma \vdash t : N \dashv \Theta'', \alpha & \text{Substituting for } \Theta' \\
☞ & \Theta; \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, N \dashv \Theta'' & \text{By Agen}
\end{array}$$

- **Case** $\dfrac{\|\Theta\|\,; \Gamma \vdash t : N}{\|\Theta\|\,; \Gamma \vdash \{t\} : \downarrow N}$ Dthunk

$$\begin{array}{rll}
& \Theta\ \mathsf{ctx} & \text{Assumption} \\
& \Theta \vdash \Gamma\,\mathsf{env} & \text{Assumption} \\
& \Theta \longrightarrow \Omega & \text{Assumption} \\
& \Omega\ \mathsf{ctx} & \text{Assumption} \\
& \Theta; \Gamma \vdash t : N \dashv \Theta' & \text{By i.h. (term size decreases)} \\
☞ & \Theta' \longrightarrow \Omega & '' \\
☞ & \Theta; \Gamma \vdash \{t\} : \downarrow N \dashv \Theta' & \text{By Athunk}
\end{array}$$

- **Case** $\dfrac{\|\Theta\|\,; \Gamma \vdash v : P}{\|\Theta\|\,; \Gamma \vdash \mathsf{return}\ v : \uparrow P}$ Dreturn

  Symmetric to Dthunk case.

- **Case** $\dfrac{\|\Theta\|\,; \Gamma \vdash v : \downarrow M \qquad \|\Theta\|\,; \Gamma \vdash s : M \gg \uparrow Q \qquad \|\Theta\| \vdash \uparrow P \leq^{-} \uparrow Q \qquad \|\Theta\|\,; \Gamma, x : P \vdash t : N}{\|\Theta\|\,; \Gamma \vdash \mathsf{let}\ x : P = v(s); t : N}$ Dambiguouslet

$$\begin{array}{rl}
\Theta\ \mathsf{ctx} & \text{Assumption} \\
\Theta \vdash \Gamma\,\mathsf{env} & \text{Assumption} \\
\Theta \longrightarrow \Omega & \text{Assumption} \\
\Omega\ \mathsf{ctx} & \text{Assumption} \\
\|\Theta\|\,; \Gamma \vdash v : \downarrow M & \text{Subderivation}
\end{array}$$

Apply the induction hypothesis to give a context Θ' such that:

$$\begin{array}{rl}
\Theta; \Gamma \vdash v : \downarrow M \dashv \Theta' & \text{By i.h. (term size decreases)}
\end{array}$$

$$\Theta' \longrightarrow \Omega \qquad ''$$

Applying well-formedness:

$$\begin{aligned}
\Theta &\longrightarrow \Theta' && \text{By Lemma K.5 (Algorithmic typing is w.f)}\\
\Theta' &\ \text{ctx} && ''\\
\Theta' \vdash \downarrow M &\ \text{type}^+ && ''\\
\downarrow M &\ \text{ground} && ''
\end{aligned}$$

Rework the second premise so we can apply the induction hypothesis:

$$\begin{aligned}
\|\Theta\|\,;\Gamma \vdash s : M \gg \uparrow Q & && \text{Premise}\\
\|\Theta\|\,;\Gamma \vdash s : [\Omega]M \gg \uparrow Q & && \text{By Lemma D.5 (Applying a context to a ground type)}\\
\|\Theta'\|\,;\Gamma \vdash s : [\Omega]M \gg \uparrow Q & && \text{By Lemma D.2 (Equality of declarative contexts)}
\end{aligned}$$

Next show the antecedents of the second premise's induction hypothesis:

$$\begin{aligned}
\Theta' &\ \text{ctx} && \text{Above}\\
\Theta &\Longrightarrow \Theta' && \text{By Lemma C.1 (}\Longrightarrow\text{ subsumes }\longrightarrow\text{)}\\
\Theta' \vdash \Gamma &\ \text{env} && \text{By Lemma C.6 (Weak context extension preserves w.f. envs)}\\
\Theta' &\longrightarrow \Omega && \text{Above}\\
\Omega &\ \text{ctx} && \text{Above}\\
\Theta' \vdash M &\ \text{type}^- && \text{Inversion (Twfshift}\downarrow\text{)}\\
[\Theta']M &= M && \text{By Lemma D.5 (Applying a context to a ground type)}
\end{aligned}$$

Apply the induction hypothesis to give a contexts Θ'', Ω' and a type Q' such that:

$$\begin{aligned}
\Theta';\Gamma \vdash s : M \gg \uparrow Q' \dashv \Theta'' & && \text{By i.h. (term size decreases)}\\
\Omega &\Longrightarrow \Omega' && ''\\
\Theta'' &\longrightarrow \Omega' && ''\\
\|\Theta'\| \vdash [\Omega']\uparrow Q' &\cong^- \uparrow Q && ''\\
[\Theta'']\uparrow Q' &= \uparrow Q' && ''\\
\Omega' &\ \text{ctx} && ''
\end{aligned}$$

Applying well-formedness:

$$\begin{aligned}
\Theta' &\Longrightarrow \Theta'' && \text{By Lemma K.5 (Algorithmic typing is w.f)}\\
\Theta'' &\ \text{ctx} && ''\\
\Theta'' \vdash \uparrow Q' &\ \text{type}^- && ''\\
[\Theta'']\uparrow Q' &= \uparrow Q' && ''
\end{aligned}$$

Now rework the third premise to match algorithmic rule. First show the third premise of the declarative rule:

$$\begin{aligned}
\|\Theta\| \vdash \uparrow P &\leq^- \uparrow Q && \text{Premise}\\
\Theta &\Longrightarrow \Theta'' && \text{By Lemma C.4 (Weak context extension is transitive)}\\
\|\Theta''\| \vdash \uparrow P &\leq^- \uparrow Q && \text{By Lemma C.3 (Equality of declarative contexts (weak))}\\
\|\Theta''\| \vdash [\Omega']\uparrow Q' &\cong^- \uparrow Q && \text{By Lemma C.3 (Equality of declarative contexts (weak))}\\
\|\Theta''\| \vdash \uparrow P &\leq^- \uparrow[\Omega']Q' && \text{By Lemma B.7 (Declarative subtyping is transitive)}\\
\|\Theta''\| \vdash P &\leq^+ [\Omega']Q' && \text{Inversion (}\leq^\pm\text{Dshift}\uparrow\text{)}
\end{aligned}$$

Show the antecedents of completeness:

$$
\begin{array}{rll}
\Theta'' & \mathsf{ctx} & \text{Above} \\
\Theta'' & \longrightarrow \Omega' & \text{Above} \\
\Omega' & \mathsf{ctx} & \text{Above} \\
\\
\Theta \vdash P & \mathsf{type}^+ & P \text{ annotation} \\
\Theta'' \vdash P & \mathsf{type}^+ & \text{By Lemma C.5 (Weak context extension preserves well-formedness)} \\
\Theta'' \vdash {\uparrow}Q' & \mathsf{type}^- & \text{Above} \\
\Theta'' \vdash Q' & \mathsf{type}^+ & \text{Inversion (Twfshift}{\uparrow}\text{)} \\
\\
P & \mathsf{ground} & P \text{ annotation} \\
[\Theta'']{\uparrow}Q' = {\uparrow}Q' & & \text{Above} \\
[\Theta'']Q' = Q' & & \text{By definition of } [-]-
\end{array}
$$

Applying Theorem G.5 (Completeness of algorithmic subtyping), we have the following for some context Θ''':

$$
\begin{array}{rl}
\Theta'' \vdash P \leq^+ Q' \dashv \Theta''' & \text{By Theorem G.5 (Completeness of algorithmic subtyping)} \\
\Theta''' \longrightarrow \Omega' & {''}
\end{array}
$$

Now appeal to the well-formedness of the algorithmic subtyping judgment:

$$
\begin{array}{rll}
\Theta''' & \mathsf{ctx} & \text{By Lemma K.5 (Algorithmic typing is w.f)} \\
\Theta'' & \longrightarrow \Theta''' & {''} \\
[\Theta''']Q' & \mathsf{ground} & {''}
\end{array}
$$

Now show the fourth premise of the declarative rule:

$$
\begin{array}{rl}
\|\Theta''\| \vdash {\uparrow}P \leq^- {\uparrow}[\Omega']Q' & \text{Above} \\
\|\Theta'''\| \vdash {\uparrow}P \leq^- {\uparrow}[\Omega']Q' & \text{By Lemma D.2 (Equality of declarative contexts)} \\
\|\Theta'''\| \vdash [\Omega']Q' \leq^+ P & \text{Inversion } (\leq^{\pm}\mathsf{Dshift}{\uparrow}) \\
\|\Theta'''\| \vdash [\Omega'][\Theta''']Q' \leq^+ [\Omega']Q' & \text{By Lemma F.4 (} \longrightarrow \text{ leads to isomorphic types)} \\
\|\Theta'''\| \vdash [\Omega'][\Theta''']Q' \leq^+ P & \text{By Lemma B.7 (Declarative subtyping is transitive)}
\end{array}
$$

Show the antecedents of completeness:

$$
\begin{array}{rll}
\Theta''' & \mathsf{ctx} & \text{Above} \\
\Theta''' & \longrightarrow \Omega' & \text{Above} \\
\Omega' & \mathsf{ctx} & \text{Above} \\
\\
\Theta''' \vdash P & \mathsf{type}^+ & \text{By Lemma D.4 (Context extension preserves w.f.)} \\
\Theta''' \vdash Q' & \mathsf{type}^+ & \text{By Lemma D.4 (Context extension preserves w.f.)} \\
\Theta''' \vdash [\Theta''']Q' & \mathsf{type}^+ & \text{By Lemma E.1 (Applying context to the type preserves w.f.)} \\
\\
[\Theta''']Q' & \mathsf{ground} & \text{Above} \\
[\Theta''']P = P & & \text{By Lemma D.5 (Applying a context to a ground type)}
\end{array}
$$

Applying Theorem G.5 (Completeness of algorithmic subtyping), we have the following for some context Θ''':

$$
\begin{array}{rl}
\Theta''' \vdash [\Theta''']Q' \leq^+ P \dashv \Theta^{(4)} & \text{By Theorem G.5 (Completeness of algorithmic subtyping)} \\
\Theta^{(4)} \longrightarrow \Omega' & {''}
\end{array}
$$

Now appeal to the well-formedness of the algorithmic subtyping judgment:

$\Theta^{(4)}$ ctx	By Lemma K.5 (Algorithmic typing is w.f)
$\Theta''' \longrightarrow \Theta^{(4)}$	″

Let the restricted context $\Theta^{(5)} = \Theta^{(4)} \upharpoonright \Theta$:

$\Theta \longrightarrow \Theta^{(5)}$	By Lemma K.1 (Well-formedness of restricted contexts)
$\Theta^{(5)}$ ctx	″
Ω ctx	Above
Ω' ctx	Above
$\Omega \Longrightarrow \Omega'$	Above
$\Omega \longrightarrow \Omega' \upharpoonright \Omega$	By Lemma K.1 (Well-formedness of restricted contexts)
$\Omega' \upharpoonright \Omega \longrightarrow \Omega$	By Lemma O.2 (Reversing context extension from a complete context)
$\Theta^{(4)} \longrightarrow \Omega'$	Above
$\Theta^{(4)} \upharpoonright \Omega \longrightarrow \Omega$	By Lemma O.3 (Pulling back restricted contexts)
$\Theta \longrightarrow \Omega$	Above
$\Theta^{(4)} \upharpoonright \Theta \longrightarrow \Omega$	By Lemma N.2 (Identical restricted contexts)
$\Theta^{(5)} \longrightarrow \Omega$	Substituting in definition of $\Theta^{(5)}$

Rework the final premise to match the algorithmic rule:

$\lVert\Theta\rVert ; \Gamma, x : P \vdash t : N$	Premise
$\lVert\Theta^{(5)}\rVert ; \Gamma, x : P \vdash t : N$	By Lemma D.2 (Equality of declarative contexts)

Show the antecedents of the final premise's induction hypothesis:

$\Theta \longrightarrow \Theta^{(5)}$	Above
$\Theta \vdash N$ type$^-$	Assumption
$\Theta^{(5)} \vdash N$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)
N ground	Assumption
$\Theta^{(5)}$ ctx	As shown above
$\Theta^{(5)} \longrightarrow \Omega$	Above
Ω ctx	Assumption
$\Theta \vdash \Gamma$ env	Assumption
$\Theta \Longrightarrow \Theta^{(5)}$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
$\Theta^{(5)} \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta \vdash P$ type$^+$	P an annotation
$\Theta^{(5)} \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
P ground	P an annotation
$\Theta^{(5)} \vdash \Gamma, x : P$ env	By Ewfvar

Apply the induction hypothesis, to give a context $\Theta^{(6)}$, such that:

☞	$\Theta^{(6)} \longrightarrow \Omega$	By i.h. (term size decreases)
	$\Theta^{(5)}; \Gamma, x : P \vdash t : N \dashv \Theta^{(6)}$	″
☞	$\Theta; \Gamma \vdash \mathsf{let}\ x : P = v(s); t : N \dashv \Theta^{(6)}$	By Aambiguouslet

- **Case**

$$\frac{\|\Theta\|\,;\Gamma \vdash v : {\downarrow}M \qquad \|\Theta\|\,;\Gamma \vdash s : M \gg {\uparrow}Q \qquad \|\Theta\|\,;\Gamma, x : Q \vdash t : N \qquad \forall P.\ \text{if } \|\Theta\|\,;\Gamma \vdash s : M \gg {\uparrow}P \text{ then } \|\Theta\| \vdash Q \cong^{+} P}{\|\Theta\|\,;\Gamma \vdash \text{let } x = v(s); t : N}\ \text{Dunambiguouslet}$$

Θ ctx	Assumption
$\Theta \vdash \Gamma$ env	Assumption
$\Theta \longrightarrow \Omega$	Assumption
Ω ctx	Assumption
$\|\Theta\|\,;\Gamma \vdash v : {\downarrow}M$	Subderivation

Apply the induction hypothesis to give a context Θ' such that:

(1)	$\Theta; \Gamma \vdash v : {\downarrow}M \dashv \Theta'$	By i.h. (term size decreases)
	$\Theta' \longrightarrow \Omega$	″

Applying well-formedness:

$\Theta \longrightarrow \Theta'$	By Lemma K.5 (Algorithmic typing is w.f)
Θ' ctx	″
$\Theta' \vdash {\downarrow}M$ type^{+}	″
${\downarrow}M$ ground	″

Rework the second premise so we can apply the induction hypothesis:

$\|\Theta\|\,;\Gamma \vdash s : M \gg {\uparrow}Q$	Premise
$\|\Theta\|\,;\Gamma \vdash s : [\Omega]M \gg {\uparrow}Q$	By Lemma D.5 (Applying a context to a ground type)
$\|\Theta'\|\,;\Gamma \vdash s : [\Omega]M \gg {\uparrow}Q$	By Lemma D.2 (Equality of declarative contexts)

Next show the antecedents of the second premise's induction hypothesis:

Θ' ctx	Above
$\Theta \Longrightarrow \Theta'$	By Lemma C.1 ($\Longrightarrow$ subsumes $\longrightarrow$)
$\Theta' \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta' \longrightarrow \Omega$	Above
Ω ctx	Above
$\Theta' \vdash M$ type^{-}	Inversion (Twfshift↓)
$[\Theta']M = M$	By Lemma D.5 (Applying a context to a ground type)

Apply the induction hypothesis to give a contexts Θ'', Ω' and a type Q' such that:

(2)	$\Theta'; \Gamma \vdash s : M \gg {\uparrow}Q' \dashv \Theta''$	By i.h. (term size decreases)
	$\Omega \Longrightarrow \Omega'$	″
	$\Theta'' \longrightarrow \Omega'$	″
(3)	$\|\Theta'\| \vdash [\Omega']{\uparrow}Q' \cong^{-} {\uparrow}Q$	″
	$[\Theta'']{\uparrow}Q' = {\uparrow}Q'$	″
	Ω' ctx	″

Applying well-formedness:

$\Theta' \Longrightarrow \Theta''$	By Lemma K.5 (Algorithmic typing is w.f)
$\text{FEV}({\uparrow}Q') \subseteq \text{FEV}(M) \cup (\text{FEV}(\Theta'') \setminus \text{FEV}(\Theta'))$	″

	Θ'' ctx	$''$
	$\Theta'' \vdash \uparrow Q'$ type$^-$	$''$
	$[\Theta'']\uparrow Q' = \uparrow Q'$	$''$

From these conclusions we can deduce:

	$FEV(Q') \subseteq FEV(M) \cup (FEV(\Theta'') \setminus FEV(\Theta'))$	By definition of FEV and above
	$FEV(M) = \emptyset$	Since M ground
	$FEV(Q') \subseteq FEV(\Theta'') \setminus FEV(\Theta')$	Substituting above equations
	$FEV(\Theta') \subseteq FEV(\Theta'')$	By Lemma O.1 (Weak context extension maintains variables)
	$FEV(Q') \cap FEV(\Theta') = \emptyset$	By definition of $\subseteq$

Next prove by contradiction that we have Q' ground from the induction hypothesis. Assume $FEV(Q') \neq \emptyset$. Then:

(4)	$\hat{\alpha} \in FEV(Q')$	Necessarily true for some $\hat{\alpha}$, otherwise Q' would not be ground

Let $R = [\Omega']\hat{\alpha}$ and define Ω'' as the complete context obtained by taking Ω' and substituting the $\hat{\alpha} = R$ context item with $\hat{\alpha} = \downarrow\uparrow R$. Now apply soundness to the algorithmic judgment but using Ω'' as the complete context:

	Θ' ctx	Above
	$\Theta' \vdash \Gamma$ env	Above
	$\Theta'' \longrightarrow \Omega''$	Since (a) $\Theta'' \longrightarrow \Omega'$ and $\hat{\alpha} \in FEV(Q')$ implies (b) $\hat{\alpha}$ is unsolved in Θ''
	$\Theta'; \Gamma \vdash s : M \gg \uparrow Q' \dashv \Theta''$	Above
	$\Omega'_L \vdash R$ type$^+$	Inversion (Cwfsolvedguess), for some prefix Ω'_L of Ω' (this is also a prefix of Ω'' by definition of Ω'')
	R ground	$''$
	$\Omega'_L \vdash \downarrow\uparrow R$ type$^+$	By Twfshift$\uparrow$ and Twfshift$\downarrow$
	$\downarrow\uparrow R$ ground	By definition of ground
	Ω'' ctx	By Ω' ctx and above two statements
	$\Theta' \vdash M$ type$^-$	Above
	$[\Theta']M = M$	Above
	$\lVert\Theta'\rVert; \Gamma \vdash s : [\Omega'']M \gg Q''$	By Theorem N.3 (Soundness of algorithmic typing) (term size decreases)
	$\lVert\Theta'\rVert \vdash Q'' \cong^+ [\Omega'']\uparrow Q'$	$''$
(5)	$\lVert\Theta\rVert \vdash Q'' \cong^+ [\Omega'']\uparrow Q'$	By Lemma D.2 (Equality of declarative contexts)
	$\Omega \Longrightarrow \Omega'$	Above
	$\Omega \Longrightarrow \Omega''$	Replacing the instance of Wcsolveguess
	$\lVert\Theta\rVert; \Gamma \vdash s : [\Omega'']M \gg Q''$	By Lemma C.3 (Equality of declarative contexts (weak))
	$\lVert\Theta\rVert; \Gamma \vdash s : M \gg Q''$	By Lemma D.5 (Applying a context to a ground type)

Now make use of the final premise:

	$\lVert\Theta\rVert \vdash Q \cong^+ Q''$	Instantiating final premise with $P = Q''$
	$\lVert\Theta\rVert \vdash Q \cong^+ [\Omega'']\uparrow Q'$	Applying Lemma B.7 (Declarative subtyping is transitive) to above and (5)
	$\lVert\Theta\rVert \vdash [\Omega']\uparrow Q' \cong^- \uparrow Q$	Applying Lemma D.2 (Equality of declarative contexts) to (3)

$\|\Theta\| \vdash [\Omega']Q' \cong^+ Q$	Inversion ($\leq^{\pm}$Dshift$\uparrow$)
$\|\Theta\| \vdash [\Omega']Q' \cong^+ [\Omega'']Q'$	By Lemma B.7 (Declarative subtyping is transitive)

However:

$\|\Theta\| \vdash R \not\cong^+ \uparrow\downarrow R$	Must have the same number of shifts on both sides of the declarative subtyping judgment
$\|\Theta\| \vdash [\Omega']Q' \not\cong^+ [\Omega'']Q'$	Since $\hat{\alpha} \in \mathrm{FEV}(Q')$ by (4)

This is a contradiction, hence Q' must be ground:

(6)	$\mathrm{FEV}(Q') = \emptyset$	Since Q' ground

Next, restrict the output context of the spine judgment:

(7)	Let $\Theta''' = \Theta'' \upharpoonright \Theta$.	
	Θ ctx	Above
	Θ'' ctx	Above
	$\Theta \Longrightarrow \Theta''$	Above
	$\Theta \longrightarrow \Theta'''$	By Lemma K.1 (Well-formedness of restricted contexts)
	Θ''' ctx	"
	Ω ctx	Above
	Ω' ctx	Above
	$\Omega \Longrightarrow \Omega'$	Above
	$\Omega \longrightarrow \Omega' \upharpoonright \Omega$	By Lemma K.1 (Well-formedness of restricted contexts)
	$\Omega' \upharpoonright \Omega \longrightarrow \Omega$	By Lemma O.2 (Reversing context extension from a complet
	$\Theta'' \longrightarrow \Omega'$	Above
	$\Theta'' \upharpoonright \Omega \longrightarrow \Omega$	By Lemma O.3 (Pulling back restricted contexts)
	$\Theta \longrightarrow \Omega$	Above
	$\Theta'' \upharpoonright \Theta \longrightarrow \Omega$	By Lemma N.2 (Identical restricted contexts)
	$\Theta''' \longrightarrow \Omega$	Substituting in definition of Θ'''

Rework the third premise to match the algorithmic judgment:

$\|\Theta\| ; \Gamma, x : Q \vdash t : N$	Premise
$\|\Theta\| \vdash [\Omega']\uparrow Q' \cong^- \uparrow Q$	Above
$\|\Theta\| ; \Gamma, x : [\Omega']Q' \vdash t : N$	By Lemma J.1 (Isomorphic environments type the same tern
$\|\Theta\| ; \Gamma, x : Q' \vdash t : N$	By Lemma D.5 (Applying a context to a ground type)
$\|\Theta'''\| ; \Gamma, x : Q' \vdash t : N$	By Lemma D.2 (Equality of declarative contexts)

Next show the antecedents of the induction hypothesis:

Θ''' ctx	Above
$\Theta \vdash \Gamma$ env	Above
$\Theta''' \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta'' \vdash Q'$ type$^+$	Inversion (Twfshift$\uparrow$)
Q' ground	Above
$\Theta''' \vdash Q'$ type$^+$	By definition of restricted context Θ''' and since Q' ground

$\Theta''' \vdash \Gamma, x : Q' \text{ env}$	By Ewfvar

$\Theta''' \longrightarrow \Omega$	Above
$\Omega \text{ ctx}$	Above
$\|\Theta'''\| ; \Gamma, x : Q' \vdash t : N$	Above
$\Theta \vdash N \text{ type}^-$	Assumption
$\Theta''' \vdash N \text{ type}^-$	By $\Theta \longrightarrow \Theta'''$ and Lemma D.4 (Context extension preserves
$N \text{ ground}$	Assumption

Applying the induction hypothesis, we have for a context $\Theta^{(4)}$:

(8)	$\Theta'''; \Gamma, x : Q' \vdash t : N \dashv \Theta^{(4)}$	By i.h. (term size decreases)
☞	$\Theta^{(4)} \longrightarrow \Omega$	″
☞	$\Theta; \Gamma \vdash \text{let } x = v(s); t : N \dashv \Theta^{(4)}$	Applying Aunambiguouslet to (1), (2), (6), (7), and (8)

- **Case**

$$\frac{}{\|\Theta\| ; \Gamma \vdash \epsilon : N \gg N} \text{Dspinenil}$$

The output context will be Θ, the complete context will be Ω, and the output type will be M.

☞	$\Theta; \Gamma \vdash \epsilon : N \gg N \dashv \Theta$	By Aspinenil
☞	$\Omega \Longrightarrow \Omega$	By Lemma C.2 (Weak context extension is reflexive)
☞	$\Theta \longrightarrow \Omega$	Assumption
	$\|\Theta\| \vdash [\Omega][\Theta]N \cong^- [\Theta]N$	By Lemma F.4 ($\longrightarrow$ leads to isomorphic types)
☞	$[\Theta]N = N$	Assumption
☞	$\|\Theta\| \vdash [\Omega]N \cong^- N$	Substituting in above equation
☞	$[\Theta]N = N$	Assumption
☞	$\Omega \text{ ctx}$	Assumption

- **Case**

$$\frac{\|\Theta\| ; \Gamma \vdash v : P \qquad \|\Theta\| \vdash P \leq^+ [\Omega]Q \qquad \|\Theta\| ; \Gamma \vdash s : [\Omega]N \gg M}{\|\Theta\| ; \Gamma \vdash v, s : [\Omega](Q \to N) \gg M} \text{Dspinecons}$$

$\Theta \text{ ctx}$	Assumption
$\Theta \vdash \Gamma \text{ env}$	Assumption
$\Theta \longrightarrow \Omega$	Assumption
$\Omega \text{ ctx}$	Assumption

By the induction hypothesis for the first premise, there exists a context Θ', such that:

$\Theta; \Gamma \vdash v : P \dashv \Theta'$	By i.h. (term size decreases)
$\Theta' \longrightarrow \Omega$	″

Apply well-formedness to this algorithmic judgment:

$\Theta \longrightarrow \Theta'$	By Lemma K.5 (Algorithmic typing is w.f)

Θ' ctx	$''$
$\Theta \vdash P$ type$^+$	$''$
P ground	$''$

Now use completeness of subtyping:

Θ' ctx	Above
$\Theta' \longrightarrow \Omega$	Above
Ω ctx	Above
$\|\Theta\| \vdash P \leq^+ [\Omega]Q$	Subderivation
$\|\Theta'\| \vdash P \leq^+ [\Omega]Q$	By Lemma D.2 (Equality of declarative contexts)
$\|\Theta'\| \vdash P \leq^+ [\Omega][\Theta']Q$	By Lemma F.4 ($\longrightarrow$ leads to isomorphic types)
$\Theta' \vdash P$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
$\Theta \vdash Q \to N$ type$^-$	Assumption
$\Theta \vdash Q$ type$^+$	By Twfarrow
$\Theta' \vdash Q$ type$^+$	By Lemma D.4 (Context extension preserves w.f.)
$\Theta' \vdash [\Theta']Q$ type$^+$	By Lemma E.1 (Applying context to the type preserves w.f.)
P ground	Above
$[\Theta'][\Theta']Q = [\Theta']Q$	By Lemma D.6 (Context application is idempotent)
$\Theta' \vdash P \leq^+ [\Theta']Q \dashv \Theta''$	By Theorem G.5 (Completeness of algorithmic subtyping)
$\Theta'' \longrightarrow \Omega$	$''$

Applying well-formedness:

Θ'' ctx	By Lemma E.2 (Algorithmic subtyping is w.f.)
$\Theta' \longrightarrow \Theta''$	$''$
$[\Theta''][\Theta']Q$ ground	$''$

Next rework the third premise to match the algorithmic rule:

$\|\Theta''\|\,; \Gamma \vdash s : [\Omega]N \gg M$	By Lemma D.2 (Equality of declarative contexts)
$\|\Theta''\| \vdash [\Omega][\Theta'']N \cong^- [\Omega]N$	By Lemma F.4 ($\longrightarrow$ leads to isomorphic types)
$\|\Theta''\|\,; \Gamma \vdash s : [\Omega][\Theta'']N \gg M'$	By Lemma J.1 (Isomorphic environments type the same terms)
$\|\Theta''\| \vdash M \cong^- M'$	$''$

Now show the antecedents of the third premise's induction hypothesis:

Θ'' ctx	Above
$\Theta \vdash \Gamma$ env	Assumption
$\Theta'' \vdash \Gamma$ env	By Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta \vdash P \to N$ type$^-$	Assumption
$\Theta \vdash N$ type$^-$	By Twfarrow
$\Theta \longrightarrow \Theta''$	By Lemma D.3 (Context extension is transitive)
$\Theta'' \vdash N$ type$^-$	By Lemma D.4 (Context extension preserves w.f.)
$\Theta'' \vdash [\Theta'']N$ type$^-$	By Lemma E.1 (Applying context to the type preserves w.f.)
$[\Theta''][\Theta'']N = [\Theta'']N$	By Lemma D.6 (Context application is idempotent)

$\Theta'' \longrightarrow \Omega$ By i.h. (term size decreases)
Ω ctx Assumption

Now apply the induction hypothesis to this third premise. This gives the following for some contexts Θ''', Ω' and type M'':

$\Theta''; \Gamma \vdash s : [\Theta'']N \gg M'' \dashv \Theta'''$ By i.h. (term size decreases)
☞ $\Omega \Longrightarrow \Omega'$ ″
☞ $\Theta''' \longrightarrow \Omega'$ ″
$\|\Theta''\| \vdash [\Omega']M'' \cong^{-} M'$ ″
☞ $[\Theta''']M'' = M''$ ″
☞ Ω' ctx ″

$\|\Theta'\| \vdash [\Omega]M'' \cong^{-} M$ By Lemma B.7 (Declarative subtyping is transitive)
☞ $\|\Theta\| \vdash [\Omega]M'' \cong^{-} M$ By Lemma D.2 (Equality of declarative contexts)

Finally, apply the algorithmic judgment:

☞ $\Theta; \Gamma \vdash v, s : P \rightarrow N \gg M'' \dashv \Theta''$ By Aspinecons

- **Case** $\dfrac{\|\Theta\| \vdash P \text{ type}^{+} \qquad \|\Theta\|\,; \Gamma \vdash s : [P/\alpha]([\Omega]N) \gg M}{\|\Theta\|\,; \Gamma \vdash s : [\Omega](\forall \alpha.\, N) \gg M}$ Dspinetypeabs

  Take cases on whether $\alpha \in \text{FUV}(N)$

  - **Case** $\alpha \notin \text{FUV}(N)$:

$[P/\alpha]([\Omega]N) = [\Omega]N$ By $\alpha \notin \text{FUV}(N)$

$\|\Theta\|\,; \Gamma \vdash s : [\Omega]N \gg M$ By equality

$\Theta \vdash \forall \alpha.\, N$ type^{-} Assumption
$\Theta, \alpha \vdash N$ type^{-} By Twfforall
$\Theta \vdash N$ type^{-} By $\alpha \notin \text{FUV}(N)$ and Lemma K.2 (Type well-formed with type variable removed)

$[\Theta](\forall \alpha.\, N) = \forall \alpha.\, N$ Assumption
$[\Theta]N = N$ By definition of $[-]-$

Θ ctx Assumption
$\Theta \vdash \Gamma$ env Assumption
$\Theta \longrightarrow \Omega$ Assumption
Ω ctx Assumption

By the induction hypothesis we have contexts Θ', Ω' and type M', such that:

☞ $\Omega \Longrightarrow \Omega'$ By i.h. (Term size stays the same and the number of prenex universal quantifiers decreases. Since applying the context only replaces positive types by positive types, it cannot change the number of prenex universal quantifiers.)
☞ $\Theta' \longrightarrow \Omega'$ ″

☞ $\|\Theta\| \vdash [\Omega]M' \cong^- M$ ″
☞ $[\Theta']M' = M'$ ″
☞ Ω' ctx ″
$\Theta; \Gamma \vdash s : N \gg M' \dashv \Theta'$ ″

Applying the algorithmic judgment:

☞ $\Theta; \Gamma \vdash s : \forall\alpha.\, N \gg M' \dashv \Theta'$ By Aspinetypeabsnotin

– **Case** $\alpha \in FUV(N)$:
First rework the premise to match the algorithmic rule:

$\|\Theta\| ; \Gamma \vdash s : [P/\alpha]([\Omega]N) \gg M$	Premise
$\|\Theta\| ; \Gamma \vdash s : [[\Omega]P/\alpha]([\Omega]N) \gg M$	P ground and Lemma D.5 (Applying a context to a ground type)
$\|\Theta\| ; \Gamma \vdash s : [\Omega]([P/\alpha]N) \gg M$	By definition of [−]−

$[\Omega, \hat{\alpha} = P](\Theta, \hat{\alpha}); [\Omega, \hat{\alpha} = P]\Gamma \vdash s : [\Omega, \hat{\alpha} = P]([\hat{\alpha}/\alpha]N) \gg M$ For fresh $\hat{\alpha}$

Now show the antecedents of the induction hypothesis:

$\Theta \vdash \forall\alpha.\, N$ type$^-$	Assumption
$\Theta, \alpha \vdash N$ type$^-$	By Twfforall
$\Theta, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]N$ type$^-$	By Lemma K.3 (Substitution preserves well-formedness of types)
$[\Theta](\forall\alpha.\, N) = \forall\alpha.\, N$	Assumption
$[\Theta]N = N$	By definition of [−]−
$[\hat{\alpha}/\alpha]([\Theta]N) = [\hat{\alpha}/\alpha]N$	By equality
$[\Theta]([\hat{\alpha}/\alpha]N) = [\hat{\alpha}/\alpha]N$	$\hat{\alpha}$ fresh
$[\Theta, \hat{\alpha}]([\hat{\alpha}/\alpha]N) = [\hat{\alpha}/\alpha]N$	By definition of [−]−
Θ ctx	Assumption
$\Theta, \hat{\alpha}$ ctx	By Cwfunsolvedguess
$\Theta \Longrightarrow \Theta$	By Lemma C.2 (Weak context extension is reflexive)
$\Theta \Longrightarrow \Theta, \hat{\alpha}$	By Wcnewunsolvedguess
$\Theta \vdash \Gamma$ env	Assumption
$\Theta, \hat{\alpha} \vdash \Gamma$ env	By $\Theta \Longrightarrow \Theta, \hat{\alpha}$ and Lemma C.6 (Weak context extension preserves w.f. envs)
$\Theta \longrightarrow \Omega$	Assumption
$\Theta, \hat{\alpha} \longrightarrow \Omega, \hat{\alpha} = P$	By Csolveguess
Ω ctx	Assumption
P ground	P declarative type
$\|\Theta\| \vdash P$ type$^+$	Premise
$\Omega \vdash P$ type$^+$	Since $\Theta \longrightarrow \Omega$, and context extension cannot add or remove universal variables
$\Omega, \hat{\alpha} = P$ ctx	By Cwfsolvedguess

Applying the induction hypothesis, we have contexts Θ', Ω' and a type M', such that:

	$\Omega, \hat{\alpha} = P \Longrightarrow \Omega'$	By i.h. (Term size stays the same and the number of prenex universal quantifiers decreases. Since applying the context only replaces positive types by positive types, it cannot change the number of prenex universal quantifiers.)
☞	$\Theta' \longrightarrow \Omega'$	″
☞	$\|\Theta\| \vdash [\Omega']M' \cong^- M$	″
☞	$[\Theta']M' = M'$	″
☞	Ω' ctx	″
	$\Theta, \hat{\alpha}; \Gamma \vdash s : [\hat{\alpha}/\alpha]N \gg M' \dashv \Theta'$	″

	$\Omega, \hat{\alpha} = P \Longrightarrow \Omega'$	Above
	$\Omega \Longrightarrow \Omega$	By Lemma C.2 (Weak context extension is reflexive)
	$\Omega \Longrightarrow \Omega, \hat{\alpha} = P$	By Wcnewsolvedguess
☞	$\Omega \Longrightarrow \Omega'$	By Lemma C.4 (Weak context extension is transitive)

Finally, applying the algorithmic judgment:

☞	$\Theta; \Gamma \vdash s : (\forall \alpha. N) \gg M' \dashv \Theta'$	By Aspinetypeabsin

□